ON BADLY APPROXIMABLE NUMBERS WHOSE APPROXIMATION SET HAS A DISCRETE DERIVATIVE

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ABSTRACT. The approximation properties of an irrational number ξ can be described by the set $A(\xi)$ which consits of all the accumulation points of $\{x: x=q(q\xi-p),\ p\in\mathbb{Z},\ q\in\mathbb{Z}\setminus\{0\},$ p and q relatively prime$\}$. First results about $A(\xi)$ were obtained by Lekkerkerker who showed that $A(\xi)$ is discrete and $0\notin A(\xi)$ if and only if ξ is a quadratic irrational. Moreover, he proved that to any compact interval I not containing zero there exists a number ξ for which $0\notin A(\xi)$ and I is contained in $A(\xi)$. In this paper we show that to any given interval I there exists an irrational ξ such that $A(\xi)$ intersected with I is countable. In addition, we give exact conditions on the continued fraction expansion of ξ for $A(\xi)'$ to be discrete and $0\notin A(\xi)$.

KEY WORDS AND PHRASES. Approximation of numbers, badly approximable numbers, continued fractions.

MATHEMATICS SUBJECT CLASSIFICATION CODES : 10 F 05, 10 F 35

INTRODUCTION

Throughout this paper, ξ denotes an irrational number which is represented by the regular continued fraction $[b_0,b_1,b_2,\ldots]$. As usual, we write A_n/B_n for the n-th convergent. Moreover, for any set A, we denote by A' and $\bar{A}$ the set of all real accumulation points of A and the closure of A, respectively, and put $|A|=\{z: z=|a|,\ a\in A\}$. The set A is called the *derived set* of A. Furthermore, for any sequence (a_n) we write $H(a_n)$ for the set of all its real limit points. Moreover, for any $\varepsilon>0$ and $x\in\mathbb{R}$ we use the abbreviation $B(x,\varepsilon):=(x-\varepsilon,x+\varepsilon)$.

A well known theorem of Dirichlet says that there are infinitely many pairs $(p,q)\in\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$ such that $|q(q\xi-p)|<1$. In order to study the approximation of ξ by rational numbers, it is therefore natural to introduce the set $A(\xi):=\{z: z=q(q\xi-p),\ p\in\mathbb{Z},\ q\in\mathbb{Z}\setminus\{0\},\ p \text{ and } q \text{ relatively prime}\}'$. Obiously, $x\in A(\xi)$ if and only if for every $\varepsilon>0$ there exist infinitely many pairs $(p,q)\in\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$ such that p and q are relatively prime and $(x-\varepsilon)q^{-2} \leq \xi-pq^{-1} \leq (x+\varepsilon)q^{-2}$. Therefore, $A(\xi)$ is called the *approximation set* of ξ , and if $0\notin A(\xi)$ then the number ξ is called *badly approximable*. A similar approach was done by Lekkerkerker [7] who investigated the set $D(\xi)$ which is the same as $A(\xi)$, except that p and q are not required to be relatively prime.

A first result concerning $A(\xi)$ which is essentially due to Lekkerkerker [7] provides a characterization of quadratic irrationals (cf. [11]):

The set $A(\xi)$ is discrete and does not contain zero if and only if ξ is a quadratic irrational.

Moreover, Lekkerkerker proved that to any compact interval I not containing 0 there is a number ξ such that $0\notin D(\xi)$ and I is contained in $D(\xi)$, a result which can easily be extended to $A(\xi)$. This suggests to ask if there are numbers ξ such that $A(\xi)$ is neither discrete nor contains an interval. In this paper we prove the existence of such numbers. Even more, we characterize all those numbers ξ for which $A(\xi)'$ is discrete and $0\notin A(\xi)$. It turns out that their regular continued fraction must belong to some subset of the class of Maillet's quasiperiodic continued fractions [8] which we call the class of pseudoperiodic continued fractions (cf. Definition 2).

The proof of this result depends essentially on the relation between $A(\xi)$ and the sequence (ξ_n) which is implicitely defined by $\xi=[b_0,b_1,\ldots,b_{n-1},\xi_n]$. In fact, it turns out that $A(\xi)'$ is discrete and $0\notin A(\xi)$ if and only if $H(\xi_n)'$ is finite and (ξ_n) is bounded. This new problem is much easier to handle because (ξ_n) can be generated by a nice iteration function τ (cf. Definition 1), whereas it seems to be rather difficult to treat $A(\xi)$ directly. The link between $A(\xi)$ and $H(\xi_n)$ is provided by the sequence $\delta_n=B_n|B_n\xi-A_n|$ which describes the approximation properties of the convergents of ξ. In [10] and [11], one of the authors used this method in order to give a short proof of Lekkerkerker's theorem and to solve related problems in diophantine approximation.

1. PRELIMINARIES

First, we state some well known formulas for continued fractions (e.g. see [5], [9]) which hold for all $n \in \mathbb{N}$:

$$\rho_n := \frac{B_n}{B_{n-1}} = [b_n, b_{n-1}, \ldots, b_1], \tag{1}$$

$$\xi - \frac{A_n}{B_n} = \frac{(-1)^n}{B_n^2(\xi_{n+1} + \frac{1}{\rho_n})}, \tag{2}$$

$$\delta_n = \frac{1}{\xi_{n+1} + \frac{1}{\rho_n}}, \tag{3}$$

$$\delta_{n-1} = \frac{1}{\rho_n + \frac{1}{\xi_{n+1}}}, \tag{4}$$

$$\sqrt{1 - 4\delta_n\delta_{n-1}} = \frac{\xi_{n+1}\rho_n - 1}{\xi_{n+1}\rho_n + 1}, \tag{5}$$

$$\xi_{n+1} = \frac{1 + \sqrt{1 - 4\delta_n\delta_{n-1}}}{2\delta_n}, \tag{6}$$

$$\frac{1}{b_{n+1} + 2} < \delta_n < \frac{1}{b_{n+1}}, \tag{7}$$

$$\delta_n \leq 1, \tag{8}$$

$$\mathrm{sign}(B_n(B_n\xi - A_n)) = (-1)^n. \tag{9}$$

The following result which is essentially due to Jurkat and Peyerimhoff [5] shows that the sequence (δ_n) is basic for evaluating $|A(\xi)|$:

Lemma 1. *Suppose ξ is represented by a continued fraction whose partial quotients are bounded by some number k. Let $\tilde{\mathbb{Z}} := \{(a,b):\ a, b \in \mathbb{Z},\ a$ and b relatively prime$\}$ and for any $c>0$ put $r_1 = r_1(c) = 2+c$ and $r_2 = r_2(c,k) = (k+1)c + 1$. Furthermore, let $f_{u,v}(x,y) = ux^2 + \sqrt{1-4uv}\,xy - vy^2$ and $H = H((\delta_n, \delta_{n-1}))$. Then we have*

$$|A(\xi)| \cap (0,c) = \Big|\bigcup_{(u,v)\in H} f_{u,v}\Big(\tilde{\mathbb{Z}} \cap (B(0,r_1) \times B(0,r_2))\Big)\Big| \cap (0,c); \tag{10}$$

$$|A(\xi)| = \Big|\bigcup_{(u,v)\in H} f_{u,v}(\tilde{\mathbb{Z}})\Big|. \tag{11}$$

The following result is essentially due to Lekkerkerker [7] .

Lemma 2. *Suppose the assumptions of Lemma 1 hold and let* $g_{u,v}(x,y) = (ux^2 + (uv-1)xy - vy^2)\cdot(uv+1)^{-1}$ *and* $\tilde{H}=H((\xi_{n+1},\rho_n))$. *Then*

$$|A(\xi)|\cap(0,c) = |\bigcup_{(u,v)\in\tilde{H}} g_{u,v}\big(\tilde{\mathbb{Z}}\cap(B(0,r_1)\times B(0,r_2))\big)|\cap(0,c)\ , \tag{12}$$

$$|A(\xi)| = |\bigcup_{(u,v)\in\tilde{H}} g_{u,v}(\tilde{\mathbb{Z}})|. \tag{13}$$

Next we introduce the τ-function which generates the sequence (ξ_n). For this we need the function $I(z)$ which stands for the greatest integer not exceeding z.

Definition 1. *Let* $\tau : \mathbb{R}\setminus\mathbb{Z} \to \mathbb{R};\ \ \tau(z) = (z - I(z))^{-1}$.

In the following figure the graph of the τ-function is shown.

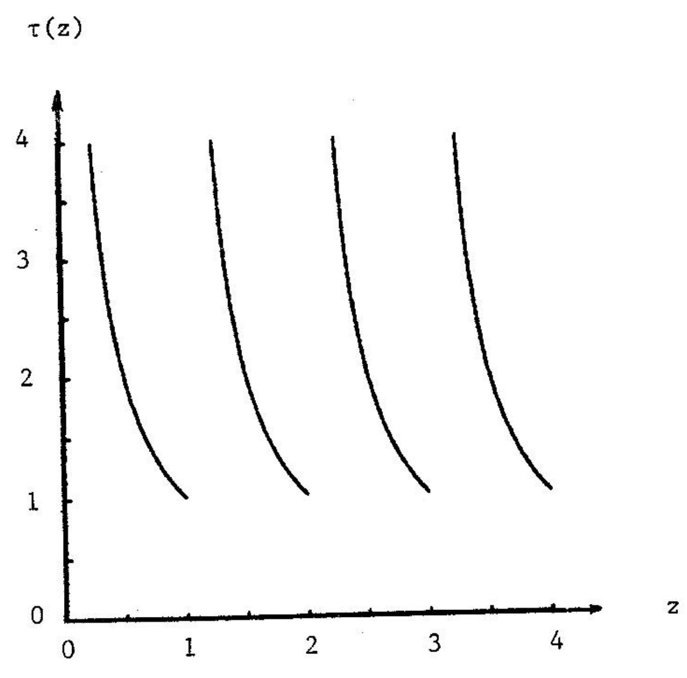

FIG. 1

The τ-function has the following properties:

(τ_1) $\xi_{n+1} = \tau(\xi_n)$ for all $n \in \mathbb{N}_o$.

(τ_2) τ is continuously differentiable on $\mathbb{R}\setminus\mathbb{Z}$.

(τ_3) τ is strictly decreasing on every connected subset of $\mathbb{R}\setminus\mathbb{Z}$.

(τ_4) If $H(\xi_n)$ is a compact subset of $\mathbb{R}\setminus\mathbb{Z}$ then $\tau(H(\xi_n)) \subset H(\xi_n)$ and $\tau(H(\xi_n)') \subset H(\xi_n)'$.

(τ_5) To every compact subset K of $\mathbb{R}\setminus\mathbb{Z}$ there exist numbers $\alpha = \alpha(K) > 1$ and $\beta = \beta(K)$ such that $\alpha \leq |\tau'(x)| \leq \beta$ for all $x \in K$.

We restrict ourselves to the proof of (τ_4). If $x \in H(\xi_n)$ then there exists a sequence (n_i) such that ξ_{n_i} converges to x. Now by (τ_1) and (τ_2) the sequence $\xi_{n_i+1} = \tau(\xi_{n_i})$ converges to $\tau(x)$ and therefore $\tau(x)$ belongs to $H(\xi_n)$. The second part of (τ_4) follows by a similar argument combined with (τ_3).

When using the τ-function a short proof of the following theorem of Ballieu [2] can be given:

The set $H(\xi_n)$ is finite and (ξ_n) is bounded if and only if ξ is represented by a periodic continued fraction.

Actually, in the next lemma we shall show even more, and the proof is a very special case of our procedure in the next section.

Lemma 3. *Suppose $t_1, t_2, \ldots, t_m \in \mathbb{R}\setminus\mathbb{Z}$ are isolated limit points of the sequence (ξ_n) such that $\tau(t_\nu) = t_{\nu+1}$ for $1 \leq \nu \leq m-1$ and $\tau(t_m) = t_1$. Then ξ is represented by a periodic continued fraction.*

Proof. Choose a number $\varepsilon > 0$ such that $\overline{B(t_\nu,\varepsilon)} \cap H(\xi_n) = \{t_\nu\}$ and $\overline{B(t_\nu,\varepsilon)} \subset \mathbb{R}\setminus\mathbb{Z}$. By ($\tau_2$) there is a number δ, $0 < \delta < \varepsilon$, such that $\tau(B(t_\nu,\delta)) \subset B(\tau(t_\nu),\varepsilon)$. Furthermore, our procedure provides a number n_o with $\xi_n \notin \bigcup_{\nu=1}^{m} \big(B(t_\nu,\varepsilon) \setminus B(t_\nu,\delta)\big)$ for all $n \geq n_o$. On the other hand there must be a $n_1 \geq n_o$ with $\xi_{n_1} \in B(t_1,\delta)$. Now by induction it is easy to see that $\xi_{n_1+\nu m+k} \in B(t_k,\delta)$. for all $\nu \in \mathbb{N}_o$ and all $k \in \{1,\ldots,m\}$. From this we conclude that the sequence $(I(\xi_{n_1+\nu}))$ is periodic and the proof is finished.

2. A NECESSARY CONDITION FOR $A(\xi)'$ TO BE DISCRETE AND $0 \notin A(\xi)$

In this section we assume that $A(\xi)'$ is discrete and $0 \notin A(\xi)$ and look for the kind of structure of the continued fraction of ξ involved by these conditions. Our first result is

Lemma 4. *Suppose that $A(\xi)'$ is discrete and $0 \notin A(\xi)$. Then $H(\xi_n)'$ is finite and (ξ_n) is bounded.*

Proof. Note first that A_n and B_n are relatively prime and that the numbers δ_k are distinct. Therefore and because of (8), $H(\delta_n)$ is a compact subset of $|A(\xi)|$, which implies that $H(\delta_n)'$ is finite and $0 \notin H(\delta_n)$.

Consider now any $x \in H(\xi_n)$. By definition of $H(\xi_n)$ there must exist a sequence (n_i) such that ξ_{n_i+1} converges to x. With the notations $F(x,y) := (2x)^{-1}(1+\sqrt{1-4xy})$ and $\Delta_n := (\delta_n, \delta_{n-1})$ we obtain from (6) that $\xi_{n_i+1} = F(\Delta_{n_i})$ for all $i \in \mathbb{N}$. Since (Δ_n) is bounded, we can assume without loss of generality that (Δ_{n_i}) converges to some $\Delta \in H(\Delta_n)$. But then

$$x = \lim_{i\to\infty} \xi_{n_i+1} = \lim_{i\to\infty} F(\Delta_{n_i}) = F(\Delta) .$$

Using the same technique, one can show that for each $y \in H(\xi_n)'$ there is some $\Gamma \in H(\Delta_n)'$ such that $y = F(\Gamma)$. Therefore, in order to finish the proof it is enough to show that $H(\Delta_n)'$ is finite.

For this we observe that in any case $H(\Delta_n)' \subset (H(\delta_n) \times H(\delta_n)') \cup (H(\delta_n)' \times H(\delta_n))$. If $H(\Delta_n)' \subset H(\delta_n)' \times H(\delta_n)'$ then we are obviously finished. In the other case, consider any $\Theta \in H(\Delta_n)' \setminus (H(\delta_n)' \times H(\delta_n)')$. There is a sequence of pairwise distinct elements $\Gamma_m \in H(\Delta_n)$ which converges to Θ. Since $\Theta \notin H(\delta_n)' \times H(\delta_n)'$ and $H(\Delta_n) \subset H(\delta_n) \times H(\delta_n)$, we can assume without loss of generality that either the fist or the second projection of the vectors Γ_m is constant in m. Now $0 \notin A(\xi)$ which implies that the partial quotients of the continued fraction of ξ are bounded. Let $f_{u,v}(x,y)$ be as in Lemma 1. Because of (4) and (8) we have $c < \delta_n \leq 1$ for some positive constant c and hence, there must exist a c_o such that $|f_{\Gamma_m}(1,1)| < c_o$ for all $m \in \mathbb{N}$. By (10) we get

$$|f_{\Gamma_m}(1,1)| \in |A(\xi)| \cap (0, c_o) \qquad \text{for all } m \in \mathbb{N}.$$

Now observe that

$$\frac{\partial^2}{\partial v^2} f_{u,v}(1,1) = \frac{-4u^2}{(1-4uv)^{3/2}} \quad \text{and} \quad \frac{\partial^2}{\partial u^2} f_{u,v}(1,1) = \frac{-4v^2}{(1-4uv)^{3/2}}$$

which shows that $f_{u,v}(1,1)$ is concave in each of the variables u and v. Therefore, and since exactly one of the projections of Γ_m is constant, we conclude that the sequence $(f_{\Gamma_m}(1,1))$ contains infinitely many distinct elements and hence $|f_\Theta(1,1)|\in|A(\xi)'|\cap(0,c_o)$. Suppose $H(\Delta_n)'$ would be infinite. Then there would exist a sequence (Θ_m) of distinct elements all lying in $H(\Delta_n)'\setminus(H(\delta_n)'\times H(\delta_n)')$ and without loss of generality exactly one of the projections of Θ_m is constant in m. By the same argument as before, the sequence $(f_{\Theta_m}(1,1))$ would contain infinitely many distinct elements. On the other side, we just proved that $|f_{\Theta_m}(1,1)|\in|A(\xi)'|\cap(0,c_o)$ for all $m\in\mathbb{N}$. Thus we get a contradiction to the hypothesis that $|A(\xi)'|\cap(0,c_o)$ is finite. This finishes the proof of Lemma 3.

For the rest of this section, we assume that $H(\xi_n)'$ is finite, the sequence (ξ_n) is bounded, and describe how these conditions affect the continued fraction of ξ. The sequence (ξ_n) is basic in this context, since it completely determines the partial denominators of the continued fraction of ξ.

The hypothesis that (ξ_n) is bounded is equivalent to the existence of a number $k\in\mathbb{N}$ such that $b_n = \overline{I(\xi_n)} \leq k$ for all $n\in\mathbb{N}$. It is well known (cf. [3]) that $\{x: x=[a_0,a_1,a_2\dots],\ |a_n|\leq k \text{ for all } n\in\mathbb{N}_o\}$ is a compact subset of $\mathbb{R}\setminus\mathbb{Z}$. Therefore, the same holds for $H(\xi_n)$, which allows us to apply (τ_4) and (τ_5) with $K=H(\xi_n)$.

Our next purpose is to describe the elements of $H(\xi_n)$ and $H(\xi_n)'$.

Lemma 5. *Assume that $H(\xi_n)'$ is a finite, nonempty set and the sequence (ξ_n) is bounded. Then*

a) Each element of $H(\xi_n)$ is a quadratic irrational.

b) To each $x\in H(\xi_n)$ there exists a $p=p(x)\in\mathbb{N}_o$ such that $\tau^{(p)}(x)\in H(\xi_n)'$ and $\tau^{(\nu)}(x)\notin H(\xi_n)'$ for $0\leq\nu\leq p-1$.

c) We have $\xi_m\notin H(\xi_n)$ for all $m\in\mathbb{N}$.

Proof. a) Consider any $x=[a_0,a_1,a_2,\ldots] \in H(\xi_n)$ and define $x_m=[a_m,a_{m+1},\ldots]$. Clearly, $x_k \in H(\xi_n)$ for all $k \in \mathbb{N}$ and the sequence (x_k) is bounded. If two elements of (x_k) coincide then the assertion follows immediately by Lagrange's theorem. In the other case we must have $H(x_k) \subset H(\xi_n)'$. Hence, $H(x_k)$ is finite and the result follows by Ballieu's theorem.
b) We just proved that the sequence (x_k) varies only on a finite set T and therefore we have $\tau(T) \subset T$ and $T \subset H(\xi_n)$. If $T \cap H(\xi_n)'$ would be empty then T consists of isolated points of $H(\xi_n)$. Without loss of generality we can assume that T satisfies the hypotheses of Lemma 3 and hence the continued fraction of ξ would be periodic. But this contradicts our assumption that $H(\xi_n)'$ is nonempty.
c) This is an easy consequence of part a) and the hypothesis that $H(\xi_n)'$ is nonempty.

Now we proceed in the following way to investigate (ξ_n): we start with some number ξ_{n_o} which is very close to an element $x_{\nu_o} \in H(\xi_n) \setminus H(\xi_n)'$. It follows from the properties of the τ-function that $\xi_{n_o+1} = \tau(\xi_{n_o})$ is very close to $\tau(x_{\nu_o}) \in H(\xi_n)$, and hence $b_{n_o+1} = I(\xi_{n_o+1}) = I(\tau(x_{\nu_o}))$. This process is repeated so often, say λ times, such that the equality

$$I(\xi_{n_o+k}) = I(\tau^{(k)}(x_{\nu_o}))$$

holds for all $k \in \{0,\ldots,\lambda-1\}$ but not for $k=\lambda$. Clearly, λ depends on the size of $|\xi_{n_o} - x_{\nu_o}|$. After this, we look for some other element $x_{\nu_1} \in H(\xi_n)$ which is very close to ξ_{n_1}, where $n_1= n_o+\lambda$. Now the whole procedure repeats over and over again and two sequences (n_i) and (ν_i) will be defined this way. We shall show that (x_{ν_i}) attains only finitely many values, and these values completely determine the structure of the continued fraction of ξ. In order to do so we have to transform our intuitive notions, such as "very close" into precise mathematical terms.

i) Let $H(\xi_n)' = \{y_1,y_2,\ldots,y_k\}$. Choose $\varepsilon>0$ such that the sets $B(y_\nu,\varepsilon)$ are pairwise disjoint and $\overline{B(y_\nu,\varepsilon)} \cap \mathbf{Z} = \emptyset$, $1 \le \nu \le k$. Moreover, we can achieve that no element of $H(\xi_n)$ lies on the boundary of one of the sets $\overline{B(y_\nu,\varepsilon)}$. Next, according to (τ_2) and (τ_4) we can find a δ with $0<\delta<\varepsilon$ and $\tau(B(y_\nu,\delta)) \subset B(\tau(y_\nu),\varepsilon)$ for all $\nu \in \{1,\ldots k\}$.

ii) Define $T = H(\xi_n) \setminus \bigcup_{\nu=1}^{k} B(y_\nu,\delta)$. Similar to the proof of Lemma 5 b) we find that T is nonempty. Furthermore, our construction makes it necessary that T is a finite set, say $T = \{t_1,t_2,\ldots,t_m\}$. Now put $S = \{x\colon x=\tau^{(\nu)}(t_\mu),\ 1\leqslant\mu\leqslant m,\ 0\leqslant\nu\leqslant p(t_\mu)-1\}$, which is a finite set, say $S\setminus T = \{t_{m+1},\ldots,t_{m+r}\}$.

iii) Since $S\cap H(\xi_n)' = \emptyset$, there is a positive number ε_o such that for $1\leqslant j\leqslant m+r$ we have $\overline{B(t_j,\varepsilon_o)}\cap H(\xi_n) = \{t_j\}$, $\overline{B(t_j,\varepsilon_o)}\cap \mathbb{Z} = \emptyset$ and either $B(t_j,\varepsilon_o)\subset V$ or $B(t_j,\varepsilon_o)\cap V = \emptyset$, where $V = \bigcup_{\nu=1}^{\kappa} B(y_\nu,\delta)$. When choosing ε_o smaller, if necessary, we can also require that the sets $B(t_j,\varepsilon_o)$ are pairwise disjoint. Now we can find a δ_o with $0<\delta_o<\varepsilon_o$ and $\tau(B(t_j,\delta_o))\subset B(\tau(t_j),\varepsilon_o)$.

iv) Finally, we abbreviate

$B(y_\nu) := B(y_\nu,\varepsilon)$ and $\tilde{B}(y_\nu) := B(y_\nu,\delta)$ for $1\leqslant\nu\leqslant k$,

$B(t_j) := B(t_j,\varepsilon_o)$ and $\tilde{B}(t_j) := B(t_j,\delta_o)$ for $1\leqslant j\leqslant m+r$,

$U := \bigcup_{j=1}^{m+r} B(t_j)$, $\tilde{U} := \bigcup_{j=1}^{m+r} \tilde{B}(t_j)$ and $\tilde{V} := \bigcup_{\nu=1}^{k} \tilde{B}(y_\nu)$.

In Figure 2 the setup is illustrated for the case k=2, m=3 and r=2.

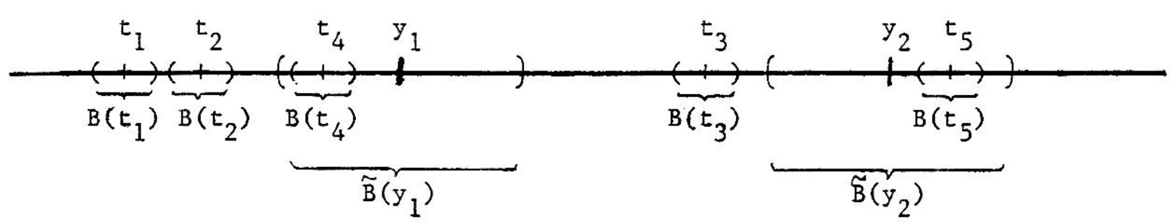

FIG. 2

We use this notation in the following lemma.

Lemma 6. *If (ξ_n) is bounded and $H(\xi_n)'$ is nonempty and finite then there are sequences (n_i), (ν_i) with*

i) $\nu_i\in\{1,2,\ldots,m\}$ *for* $i\in\mathbb{N}_o$,

ii) $\xi_{n_i}\in B(t_{\nu_i})$ *for* $i\in\mathbb{N}_o$,

iii) $\xi_{n_i+q}\in B(\tau^{(q)}(t_{\nu_i}))$ *for all q satisfying* $0\leqslant q< n_{i+1}-n_i$,

iv) $I(\xi_{n_i+q}) = I(\tau^{(q)}(t_{\nu_i}))$ *for all q satisfying* $0\leqslant q< n_{i+1}-n_i$,

v) $n_{i+1}\geqslant n_i$, $i\in\mathbb{N}_o$, *and* $\lim_{i\to\infty}(n_{i+1}-n_i) = \infty$.

Proof. We shall introduce the sequences (n_i), (ν_i) by induction.
Our construction ensures the existence of a number n_o such that $\xi_n \in V \cup \tilde{U}$ for all $n \geq n_o$. When choosing n_o bigger, if necessary, we can also require that $\xi_{n_o} \in \tilde{B}(t_{\nu_o})$ for some $\nu_o \in \{1,\ldots,m\}$.
Suppose now that we have already found n_i, ν_i with $\xi_{n_i} \in B(t_{\nu_i})$ and $\nu_i \in \{1,\ldots,m\}$. Put $n_{i+1} = 1 + \max\{n\colon \tau^{(p)}(\xi_{n_i}) \in B(\tau^{(p)}(t_{\nu_i})),\ 0 \leq p \leq n-n_i\}$. This maximum exists, since otherwise we get a contradiction to our hypothesis that $H(\xi_n)'$ is nonempty. Hence, n_{i+1} is well defined and our construction ensures that $\xi_{n_{i+1}} = \tau^{(n_{i+1}-n_i)}(\xi_{n_i}) \in B(x) \setminus \tilde{B}(x)$, where $x = \tau^{(n_{i+1}-n_i)}(t_{\nu_i})$. Since $\xi_{n_i+1} \notin U \setminus \tilde{U}$, we must have $x \in H(\xi_n)'$. Hence $\xi_{n_i+1} \notin V$ and since $\xi_n \in V \cup \tilde{U}$ for all $n \geq n_o$ there exists a unique $p \in \{1,\ldots,m\}$ with $\xi_{n_i+1} \in \tilde{B}(t_p)$. Now we put $\nu_{i+1} = p$ and the sequences (n_i), (ν_i) are well defined. Moreover, it is clear that these sequences satisfy i)-iv).
It remains to show that v) is fulfilled, too. There is a compact set $K \subset \mathbb{R} \setminus \mathbb{Z}$ such that $K \supseteq V \cup \tilde{U}$. Thus, according to (τ_5), there is a $\beta > 1$ such that $|\tau'(x)| \leq \beta$ for all $x \in V \cup \tilde{U}$. Now by v) we can successively apply the mean value theorem and obtain

$$|\xi_{n_i+v} - \tau^{(v)}(t_{\nu_i})| = |\tau^{(v)}(\xi_{n_i}) - \tau^{(v)}(t_{\nu_i})| \leq \beta|\tau^{(v-1)}(\xi_{n_i}) - \tau^{(v-1)}(t_{\nu_i})| \leq \beta^v|\xi_{n_i} - t_{\nu_i}|$$

for $v \in \{0,\ldots,n_{i+1}-n_i\}$. The right hand side of these inequalities is unequal to zero according to Lemma 5 c), hence, for $v = n_{i+1}-n_i$ we get

$$n_{i+1} - n_i \geq \frac{1}{\log(\beta)} \log\left(\frac{|\xi_{n_{i+1}} - \tau^{(n_{i+1}-n_i)}(t_{\nu_i})|}{|\xi_{n_i} - t_{\nu_i}|}\right) \geq \frac{1}{\log(\beta)} \log\left(\frac{\min(\delta,\delta_o)}{|\xi_{n_i} - t_{\nu_i}|}\right). \tag{14}$$

Recall now that the elements of T are isolated limit points, hence $|\xi_{n_i} - t_{\nu_i}|$ converges to zero, and i) follows from (14). This finishes the proof of Lemma 6.

Remarks. 1) The proof of v) required a lower bound for $n_{i+1}-n_i$. An upper bound can also be obtained in quite the same way, namely

$$n_{i+1} - n_i \leq \frac{1}{\log(\alpha)} \log\left(\frac{\max(\delta,\delta_o)}{|\xi_{n_i} - t_{\nu_i}|}\right) \tag{15}$$

where $\alpha = \alpha(K)$ as defined in (τ_5).

2) Note that the conclusion of Lemma 6 remains valid if we replace (ξ_n) by an abitrary iteration sequence (a_n) which is generated by a function f satisfying $|f'(x)|>1$ for all $x \in \overline{\{a_n : n \in \mathbb{N}\}}$.

We are now in the position to describe the continued fraction of ξ. It follows immediately from iv) that the first $n_{i+1}-n_i$ partial quotients of the continued fraction of ξ_{n_i} and t_{ν_i} coincide. According to Lemma 5 and Lagrange's theorem, each number t_ν can be represented by a periodic continued fraction, say $t_\nu = [a_0^{(\nu)},\dots,a_{p_\nu}^{(\nu)},\overline{s_1^{(\nu)},\dots,s_{k_\nu}^{(\nu)}}]$, and therefore the continued fraction of ξ is built up by larger and larger pieces of periodic continued fractions. This suggests the following definition.

Definition 2. *The continued fraction of* ξ *is called pseudoperiodic if and only if there are periodic continued fractions* $t_\nu = [a_0^{(\nu)},\dots,a_{p_\nu}^{(\nu)},\overline{s_1^{(\nu)},\dots,s_{k_\nu}^{(\nu)}}]$, $1\le\nu\le m$, *and sequences* (ν_i), (λ_i) *and some* $n_0 \in \mathbb{N}$ *such that*

i) $1\le\nu_i\le m$ *and* $H(\nu_i) = \{1,2,\dots,m\}$,

ii) $\lambda_i \in \mathbb{N}$ *and* $\lim_{i\to\infty} \lambda_i = \infty$,

iii) $$\xi_{n_0} = [a_0^{(\nu_0)},\dots,a_{p_{\nu_0}}^{(\nu_0)},\underbrace{s_1^{(\nu_0)},\dots,s_{k_{\nu_0}}^{(\nu_0)}}_{\lambda_0},a_0^{(\nu_1)},\dots,a_{p_{\nu_1}}^{(\nu_1)},\underbrace{s_1^{(\nu_1)},\dots,s_{k_{\nu_1}}^{(\nu_1)}}_{\lambda_1},\dots]$$
$$= [a_0^{(\nu_i)},\dots,a_{p_{\nu_i}}^{(\nu_i)},\underbrace{s_1^{(\nu_i)},\dots,s_{k_{\nu_i}}^{(\nu_i)}}_{\lambda_i}]_{i=0}^{\infty},$$

where $\underbrace{s_1^{(\nu)},\dots,s_{k_\nu}^{(\nu)}}_{\lambda}$ *stands for the* λ*-fold repetition of* $s_1^{(\nu)},\dots,s_{k_\nu}^{(\nu)}$.

Remark. Clearly, every periodic continued fraction is pseudoperiodic and every pseudoperiod continued fraction is quasiperiodic in the sense of Maillet [8].

We are now in the position to prove the main result of this section.

Lemma 7. *Suppose* $A(\xi)'$ *is discrete and* $0 \notin A(\xi)$. *Then* ξ *is represented by a pseudoperiodic continued fraction.*

Proof. If $A(\xi)' = \emptyset$ then the result is trivial according to Lekkerkerker's theorem. In the other case let (n_i), (ν_i) and t_ν be as in Lemma 6. Without loss of generality we can assume that

$H(\nu_i) = \{1,2,\ldots,m\}$. Also, since the continued fraction of t_ν can also be written in the form $t_\nu = [a_0^{(\nu)},\ldots,a_{p_\nu+r}^{(\nu)},s_{r+1}^{(\nu)},s_{r+2}^{(\nu)},\ldots s_{k_\nu}^{(\nu)},s_1^{(\nu)},\ldots,s_r^{(\nu)}]$, $0\leq r<k_\nu$, there is no loss in generality if we assume that $I\left(\tau^{(p_{\nu_i}+\lambda_i)}(t_{\nu_i})\right) = s_{k_{\nu_i}}$ all $i\in \mathbb{N}_0$, where $\lambda_i := n_{i+1}-n_i$. Now the result follows immediately by Lemma 6.

3. THE CONVERSE OF LEMMA 7

In this section we consider an arbitrary irrational ξ which is represented by a pseudoperiodic continued fraction and evaluate $A(\xi)$. It turns out that $A(\xi)'$ is discrete and $0\notin A(\xi)$ which is the converse of Lemma 7.

First, we shall show that $0\notin A(\xi)$. From the fact that the partial quotients of any pseudoperiodic continued fraction are bounded we deduce by (7) that $0\notin H(\delta_n)$. Furthermore, it is well known (e.g. cf. [2]) that the inequality $|q\xi-p| < |B_n\xi-A_n|$ for $(p,q)\in \mathbb{Z}\times\mathbb{N}$ implies $q>B_n$. From this it follows that $\min |A(\xi)| = \min H(\delta_n)$ and hence $0\notin A(\xi)$.

The fact that the partial quotients of any pseudoperiodic continued fraction are bounded is also important for the proof that $A(\xi)'$ is discrete. Indeed, $A(\xi)$ can be evaluated explicitely by means of Lemma 2. But this makes it necessary to investigate the sequence (Θ_n) defined by $\Theta_n:=(\xi_{n+1},\rho_n)$. For this, we need the following lemma.

Lemma 8. *Let F_n denote the n-th Fibonacci number and suppose $x = [a_0,a_1,\ldots,a_n,a_{n+1},a_{n+2},\ldots]$ and $y = [a_0,a_1,\ldots,a_n,a'_{n+1},a'_{n+2},\ldots]$. Then $|x-y| \leq F_n^{-2}$.*

Proof. The special properties of x and y imply that they have the n-th convergent $\frac{A_n}{B_n}$ in common. Therefore, it follows from (2) that

$$|x-y| = \left|(x-\frac{A_n}{B_n}) - (y-\frac{A_n}{B_n})\right| \leq \max\{|x-\frac{A_n}{B_n}|,\ |y-\frac{A_n}{B_n}|\} \leq B_n^{-2} .$$

Now $B_n = B_n(x)$ is smallest if we put $x = [\bar{1}]$, but in this case (B_n) satisfies the recursive formula $B_n = B_{n-1} + B_{n-2}$, $n\geq 1$, and $B_{-1}=0$ and $B_0=1$, which is a generating system for the Fibonacci numbers. This finishes the proof of Lemma 8.

Suppose that the continued fraction of ξ is pseudoperiodic, say

$\xi_{n_o} = \Big[a_0^{(\nu_i)},\dots,a_{p_{\nu_i}}^{(\nu_i)},\underbrace{s_1^{(\nu_i)},\dots,s_{k_{\nu_i}}^{(\nu_i)}}_{\lambda_i}\Big]_{i=0}^{\infty}$. For $1\leq \alpha,\ \beta \leq m$, $\lambda\in\mathbb{N}_o$ and suitable r define

$$x_{r,\lambda}^{(\alpha,\beta)} := \Big(\big[s_r^{(\alpha)},\dots,s_{k_\alpha}^{(\alpha)},\overline{s_1^{(\alpha)},\dots,s_{k_\alpha}^{(\alpha)}}\big],\big[s_{r-1}^{(\alpha)},\dots,s_1^{(\alpha)},\underbrace{s_{k_\alpha}^{(\alpha)},\dots,s_1^{(\alpha)}}_{\lambda},a_{p_\beta}^{(\beta)},\dots,a_0^{(\beta)},\overline{s_{k_\beta}^{(\beta)},\dots,s_1^{(\beta)}}\big]\Big)$$

$$y_{r,\lambda}^{(\alpha,\beta)} := \Big(\big[s_r^{(\beta)},\dots,s_{k_\beta}^{(\beta)},\underbrace{s_1^{(\beta)},\dots,s_{k_\beta}^{(\beta)}}_{\lambda},a_0^{(\alpha)},\dots,a_{p_\alpha}^{(\alpha)},\overline{s_1^{(\alpha)},\dots,s_{k_\alpha}^{(\alpha)}}\big],\big[s_{r-1}^{(\beta)},\dots,s_1^{(\beta)},\overline{s_{k_\beta}^{(\beta)},\dots,s_1^{(\beta)}}\big]\Big)$$

$$x_r^{(\alpha)} := \Big(\big[s_r^{(\alpha)},\dots,s_{k_\alpha}^{(\alpha)},\overline{s_1^{(\alpha)},\dots,s_{k_\alpha}^{(\alpha)}}\big],\big[s_{r-1}^{(\alpha)},\dots,s_1^{(\alpha)},\overline{s_{k_\alpha}^{(\alpha)},\dots,s_1^{(\alpha)}}\big]\Big)$$

$$u_r^{(\alpha,\beta)} := \Big(\big[a_r^{(\alpha)},\dots,a_{p_\alpha}^{(\alpha)},\overline{s_1^{(\alpha)},\dots,s_{k_\alpha}^{(\alpha)}}\big],\big[a_{r-1}^{(\alpha)},\dots,a_0^{(\alpha)},\overline{s_{k_\beta}^{(\beta)},\dots,s_1^{(\beta)}}\big]\Big) .$$

We use this notation in the next lemma the proof of which is not difficult but rather tedious.

Lemma 9. *Suppose that the continued fraction of ξ is pseudoperiodic but not periodic and define $T = H(\nu_i,\nu_{i-1})$. Then*

a) $H(\Theta_n) = \{x_{r,\lambda}^{(\alpha,\beta)} : (\alpha,\beta)\in T,\ 1\leq r\leq k_\alpha,\ \lambda\in\mathbb{N}_o\} \cup \{y_{r,\lambda}^{(\alpha,\beta)} : (\alpha,\beta)\in T,\ 1\leq r\leq k_\beta,\ \lambda\in\mathbb{N}_o\}$

$\cup\ \{x_r^{(\alpha)} : 1\leq\alpha\leq m,\ 1\leq r\leq k_\alpha\} \cup \{u_r^{(\alpha,\beta)} : (\alpha,\beta)\in T,\ 0\leq r\leq p_\alpha\}$,

b) $H(\Theta_n)' = \{x_r^{(\alpha)} : 1\leq\alpha\leq m,\ 1\leq r\leq k_\alpha\}$,

c) The set $H(\xi_n)'$ *is finite and* (ξ_n) *is bounded.*

Proof. a) We define a sequence (μ_i) by $\mu_o := n_o$ and $\mu_{i+1} := \mu_i + q_{\nu_i} + \lambda_i k_{\nu_i} + 1$. Roughly speaking, (μ_i) indicates the boundaries of two adjacent blocks of the continued fraction of ξ. Furthermore, if $(\alpha,\beta)\in T$, there is a sequence (q_i) such that $(\mu_{q_i},\mu_{q_i-1}) = (\alpha,\beta)$ for all $i\in\mathbb{N}_o$ and $\lim\limits_{i\to\infty} q_i = \infty$. Now by Lemma 8 we get

$$\lim_{i\to\infty} \theta_{\mu_{q_j}+r} = u_r^{(\alpha,\beta)} \quad , 0\leq r\leq p_\alpha ;$$

$$\lim_{i\to\infty} \theta_{\mu_{q_i}+\lambda k_\alpha+p_\alpha+r} = x_{r,\lambda}^{(\alpha,\beta)} \quad , 1\leq r\leq k_\alpha,\ \lambda\in\mathbb{N}_o ;$$

$$\lim_{i\to\infty} \theta_{\mu_{q_i}-(\lambda+1)k_\beta+r-1} = y_{r,\lambda}^{(\alpha,\beta)} \quad , 1\leq r\leq k_\beta,\ \lambda\in\mathbb{N}_o .$$

Moreover, *if* (v_{q_i}) is any sequence with $\lim\limits_{i\to\infty} v_{q_i} = \lim\limits_{i\to\infty}(\lambda_{q_i} - v_{q_i}) = \infty$, then

$$\lim_{i\to\infty} \theta_{\mu_{q_i}+v_{q_i}k_\alpha+p_\alpha+r} = x_r^{(\alpha)} \quad , 1\leq r\leq k_\alpha .$$

From this we can see that the union of the sets on the right hand side of the equation in a) is contained in $H(\Theta_n)$. But the other inclusion holds as well, since a subsequence of any convergent sequence (Θ_{s_i}) with $\lim_{i\to\infty} s_i = \infty$ must satisfy one of the four cases considered above.

b) The hypothesis that ξ is nonperiodic ensures that for any $x_q^{(\gamma)}$ there exist $(\alpha,\beta)\in T$ and $r\in\{1,\dots,k_\alpha\}$ such that $x_q^{(\gamma)} = x_r^{(\alpha)}$ and the set $\{x_{r,\lambda}^{(\alpha,\beta)} : \lambda\in\mathbb{N}_0\}$ consists of infinitely many distinct elements. Now we observe that $\{x_r^{(\alpha)}\} = \{x_{r,\lambda}^{(\alpha,\beta)} : \lambda\in\mathbb{N}_0\}'$ and the assertion in b) is proved.

c) This is an easy consequence of b) and the fact that the partial quotients of pseudoperiodic continued fractions are bounded.

Remarks. 1) Suppose that the continued fraction of ξ is pseudoperiodic but not periodic. Then it follows from Lemma 9 that for any $(u,v)\in H((\xi_{n+1},\rho_n))$ the numbers u and v are quadratic irrationals. With the notation as in Lemma 1 and Lemma 2 we obtain that $g_{u,v}(\tilde{\mathbb{Z}})$ is a set of algebraic numbers of an even degree not exceeding four. According to (13) the same holds for $A(\xi)$. Although the degree 4 is attained in most cases, examples like $\xi = [\underbrace{1,\dots,1}_{\lambda},2]_{\lambda=1}^{\infty}$ show that this need not always be the case.

2) If ξ is represented by a pseudoperiodic continued fraction which is not periodic then with the notation as in Definition 2 we have that $A(\xi)$ depends only on the numbers $t_1,\dots,t_m$ and the set $H((\nu_i,\nu_{i-1}))$. In particular, $A(\xi)$ does not depend on the sequence (λ_i) and the detailed properties of the sequence (ν_i,ν_{i-1}). Now by changing the sequence (λ_i) we see that there are uncountably many irrational numbers η such that $A(\xi) = A(\eta)$. In particular, we have that $\min|A(\xi)| = \min|A(\eta)|$ for uncountably many irrational numbers η, a result which is interesting for the investigation of the Lagrange spectrum $L = \{z: z=\min|A(\xi)|, \xi\in\mathbb{R}\setminus\mathbb{Q}\}$ (cf. for example [4],[12]).

3) Let ξ satisfy the hypothesis of Lemma 9 and let t_ν be as in Definition 2. Define $t_{\nu,n} = \tau^{(n)}(t_\nu)$, $1\le\nu\le m$, $n\in\mathbb{N}_0$, then it follows from Lemma 9 b) that

$$H(\xi_n)' = \bigcup_{\nu=1}^{m} H(t_{\nu,n}) , \tag{16}$$

i.e. $H(\xi_n)'$ depends only on the numbers t_ν.

Next we prove the converse of Lemma 7.

Lemma 10. *Suppose ξ is represented by a pseudoperiodic continued fraction. Then $A(\xi)'$ is discrete and $0 \notin A(\xi)$.*

Proof. In the beginning of this section we already proved that $0 \notin A(\xi)$. Hence, it just remains to show that $A(\xi)'$ is discrete. If the continued fraction of ξ is periodic then this is tivially true. In the other case Lemma 2 gives us for all $c>0$

$$|A(\xi)'| \cap (0,c) \subset \bigcup_{(u,v)\in\tilde{H}'} |g_{u,v}(\mathbb{Z} \cap (B(0,r_1)\times B(0,r_2))| \cap (0,c) \quad . \tag{17}$$

But according to Lemma 9b, $\tilde{H}' = H(\xi_{n+1},\rho_n)'$ is finite and the proof is finished.

4. THE MAIN RESULTS

Now we are in the position to state the main results of this paper. We start with a generalization of Ballieu's theorem which follows immediately from Lemma 9c and the results of in the second section.

Theorem 1. *The set $H(\xi_n)'$ is finite and (ξ_n) is bounded if and only if ξ is represented by a pseudoperiodic continued fraction.*

When combining Lemma 7 and Lemma 10 we obtain the most important theorem in this paper.

Theorem 2. *The set $A(\xi)'$ is discrete and $0 \notin A(\xi)$ if and only if ξ is represented by a pseudoperiodic continued fraction.*

Remark. By Lekkerkerker's theorem it is obvious that $A(\xi)'$ is a nonempty and discrete set if and only if ξ is represented by a nonperiodic pseudoperiodic continued fraction.

Formula (6) and Theorem 2 also imply the following extension of a theorem of Jurkat and Peyerimhoff [5]:

Corollary 1. *The set $H(\delta_n)'$ is discrete and $0 \notin H(\delta_n)$ if and only if the continued fraction of ξ is pseudoperiodic.*

In [5], for any irrational number ξ the sequence $(n\|n\xi\|)$ is considered, where $\|x\| = \min\{|x-k| : k\in\mathbb{Z}\}$ and n ranges over $\mathbb{N}$. The following result is proved in this paper. *The set $H(n\|n\xi\|)$ is discrete and does not contain zero if and only if ξ is represented by a periodic continued fraction.*

Now an easy reflection shows that $H(n\|n\xi\|) = |A(\xi)|$. This and Theorem 2 give us the following result.

Corollary 2. *The set $H(n\|n\xi\|)'$ is discrete and $0 \notin H(n\|n\xi\|)$ if and only if ξ is represented by a pseudoperiodic continued fraction.*

Next, we want to get more information about $|A(\xi)'|$. So far, we only know a set which contains $|A(\xi)'|$ but we have no way of how to compute $|A(\xi)'|$. Our next theorem fills this gap and gives complete answer about $|A(\xi)'|$ (compare (16) for an analogous result on $H(\xi_n)'$).

Theorem 3. *Suppose ξ is represented by a nonperiodic pseudoperiodic continued fraction. Let $t^{(\mu)}$, $1 \leq \mu \leq m$ be as in Definition 2. Then*

$$|A(\xi)'| = \bigcup_{\mu=1}^{m} |A(t^{(\mu)})| \, . \tag{18}$$

Proof. We use the notation of Lemma 9. It follows from (13) that

$$|A(t^{(\mu)})| = \bigcup_{r=1}^{k_\mu} |g_{x_r^{(\mu)}}(\tilde{\mathbb{Z}})| \tag{19}$$

for all $\mu \in \{1,\dots,m\}$. This together with (17) implies

$$|A(\xi)'| \subset \bigcup_{\mu=1}^{m} |A(t^{(\mu)})| \, .$$

In order to prove the other inclusion consider any $z \in \bigcup_{\mu=1}^{m} |A(t^{(\mu)})|$. Now $z = |g_{x_p^{(\mu)}}(a,b)|$ for some $x_p^{(\mu)} \in H(\theta_n)'$ and $(a,b) \in \tilde{\mathbb{Z}}$. Now there exist $(\alpha,\beta) \in H(\mu_i,\mu_{i-1})$ and $r \in \{1,\dots,k_\alpha\}$ such that $x_p^{(\mu)} = x_r^{(\alpha)}$ and the set $\{x_{r,s}^{(\alpha,\beta)} : s \in \mathbb{N}_0\}$ is infinite.

Since $x_{r,s}^{(\alpha,\beta)} \in H(\theta_n)$ the numbers $z_s := |g_{x_{r,s}^{(\alpha,\beta)}}(a,b)|$ belong to $|A(\xi)|$ for all $s \in \mathbb{N}_0$. Moreover, the sequence (z_s) converges to z because $x_p^{(\mu)} = x_r^{(\alpha)} = \lim_{s\to\infty} x_{r,s}^{(\alpha,\beta)}$. The only thing that remains to show is that the sequence (z_s) assumes infinitely many values. Observe that the first projection of the vectors $x_{r,s}^{(\alpha,\beta)}$, $s \in \mathbb{N}_0$, is constant. Therefore, the second projection must assume infinitely many values and $z \in |A(\xi)'|$. This completes the proof of Theorem 3.

Remarks. 1) Now the question arises if the equation (18) is still valid when replacing $|A(\xi)'|$ and $|A(t^{(\mu)})|$ by $A(\xi)'$ and $A(t^{(\mu)})$, respectively. We shall construct a counter-example which shows that this is not true. For this we need the following well known theorem of Légendre [6]: Suppose $y \in \mathbb{R} \setminus \mathbb{Q}$ then each irreducible fraction p/q with $|q(qy-p)| \leq \frac{1}{2}$ is a convergent of the continued fraction of y.

If we put $\Delta_n(y) = B_n(B_n y - A_n)$ (in this case A_n/B_n denotes the n-th convergent of y), we obtain for every $c \in (0,\frac{1}{2}]$

$$A(y) \cap (-c,c) = H(\Delta_n(y)) \cap [-c,c] \ ; \tag{20}$$

$$A(y)' \cap (-c,c) = H(\Delta_n(y))' \cap [-c,c] \ . \tag{21}$$

Furthermore, by (9) we get $\Delta_n(y) = (-1)^n \delta_n(y)$.

Let $\xi = [1,1,\underbrace{5,1}_{i}]_{i=1}^{\infty} = [1,1,5,1,1,1,5,1,5,1,1,1,5,1,5,1,5,1,1,1,5,1,\ldots]$. Then ξ is pseudoperiodic and we can choose m=1 and $t=t^{(1)} = [1,1,\overline{5,1}]$. Define $a_1 = \left([\overline{5,1}] + \frac{1}{[\overline{1,5}]}\right)^{-1}$ and $a_2 = \left([\overline{1,5}] + \frac{1}{[\overline{5,1}]}\right)^{-1}$. Now by (1) and (4) we have $H(\delta_n(t)) = \{a_1,a_2\}$, $H(\Delta_n(t)) = \{-a_1,a_2\}$ and $H(\Delta_n(\xi))' = \{a_1,-a_2\}$. Clearly $a_1 < \frac{1}{5}$ and $a_2 > \frac{1}{3}$. Hence by (20), (21) we get $A(t) \cap (-\frac{1}{5},\frac{1}{5}) = \{a_1\}$ and $A(\xi)' \cap (-\frac{1}{5},\frac{1}{5}) = \{-a_1\}$ and the disproof is finished.

2) There is a companion of the equation (16). In fact, with the notation $\delta_n(y) = B_n(y)|B_n(y)y - A_n(y)|$, it is not hard to see from (1) and (4) that

$$H(\delta_n(\xi))' = \bigcup_{\mu=1}^{m} H(\delta_n(t^{(\mu)})) \ , \tag{22}$$

provided that ξ satisfies the hypothesis of Theorem 3.

3) For any open interval $I \subset \mathbb{R}$ there exists a number ξ such that $A(\xi) \cap I$ is countably infinite. For the proof of this we use the function g which is defined in Lemma 2. Clearly, there exists a $k \in \mathbb{N}$ such that $I \subset (-k,k)$. Now we can find an irrational number u, $u > 1$, such that

$$h(u) := g_{u,u}(k,k+1) = \frac{(u^2-1)k(k+1) - (2k+1)u}{(u^2+1)} \in I \ .$$

Suppose that $u = [a_0,a_1,a_2,\ldots]$ then, since the function h is continuous, there exists a $\nu \in \mathbb{N}$ such that the number $\bar{\xi} = [\overline{a_0,\ldots,a_{\nu-1},a_\nu,a_\nu,a_{\nu-1},\ldots,a_0}]$ satisfies the relation $h(\bar{\xi}) \in I$. Now from (1) we can see that $\lim_{i\to\infty} (\bar{\xi}_{2(\nu+1)i}, \bar{\rho}_{2(\nu+1)i-1}) = (\bar{\xi},\bar{\xi})$. Therefore, we have that $(\bar{\xi},\bar{\xi}) \in H((\bar{\xi}_{n+1},\bar{\rho}_n))$ and from Lemma 2 it follows that $|h(\bar{\xi})| \in |A(\bar{\xi})|$. Hence, we have $h(\bar{\xi}) \in A(\bar{\xi}) \cup A(-\bar{\xi})$, and since $h(\bar{\xi}) \in I$, we can assume without loss of generality that $I \cap A(\bar{\xi}) \neq \emptyset$. Now choose $b \in \mathbb{N}$ such that $b \notin \{a_0,\ldots,a_\nu\}$ and put $\xi = [b,\underbrace{a_0,\ldots,a_\nu,a_\nu,\ldots a_0}_{i}]_{i=0}^{\infty}$ then our conclusion follows from Theorem 3.

The assertions of Theorems 2 and 3 become plausible by the following point of view: Recall that two numbers α and β are equivalent if and only if their continued fractions can be written in the forms $\alpha = [a_0,\dots,a_k,c_0,c_1,\dots]$ and $\beta = [b_0,\dots,b_j,c_0,c_1,\dots]$. Obviously, Lemma 1 implies that $|A(\alpha)| = |A(\beta)|$ if α and β are equivalent. Suppose that the continued fraction of ξ is pseudoperiodic but not periodic. The sequence (ξ_n) gets over and over again arbitrarily close to some numbers η which are equivalent to the numbers $t^{(\mu)}$, $1 \leq \mu \leq m$, and these are the only numbers which are approximated in this way. Thus, roughly speaking, ξ is almost equivalent to each of the numbers $t^{(\mu)}$, but to no number of another equivalence class. Therefore, the approximation properties of ξ are mainly determined by the approximation properties of the numbers $t^{(\mu)}$. Moreover, the infinitely

Moreover, the infinitely many elements of $A(\xi)$ in each neighborhood of any element of $A(t^{(\mu)})$ are due to the slow transition of from the influence of $t^{(\mu)}$ to that of $t^{(\nu)}$, where $(\mu,\nu) \in H((\mu_i,\mu_{i-1}))$.

The fact that ξ can be very well approximated by quadratic irrationals together with additional growth conditions on (λ_i) implies that ξ is transcendental. Investigations like this were done for numbers whose continued fractions belong to the larger class of quasiperiodic continued fractions. Results were obtained by Maillet [8] who used elementary methods and Baher [1] who made use of a well known theorem of W. Schmidt [13]. Baher found the following theorem.

Suppose that ξ is quasiperiodic, say $\xi = [\underbrace{a_{n_i},\dots,a_{n_i+k_i-1}}_{\lambda_i}]_{i=1}^{\infty}$, and $0 \notin A(\xi)$. Then ξ is transcendental if

$$\lim_{i\to\infty} \sup\left(\frac{\lambda_i}{n_i}\right) > B$$

where B is a constant depending on the upper bound of the sequence (a_n).

Thus, we can explicitely write down many pseudoperiodic continued fractions which represent transcendental numbers.

5. FINAL REMARKS

In this last section we discuss some generalizations of the results in this paper.

1. The set $D(\xi)$.

So far we were basicly concerned about the set $A(\xi)$. However, Theorems 1 and 2 still remain valid if we replace $A(\xi)$ and $A(t^{(\mu)})$ by $D(\xi)$ and $D(t^{(\mu)})$, respectively. The reason why this holds is that the equations (10), (11) are also true if we write $D(\xi)$ and $(\mathbb{Z}\times\mathbb{Z})\setminus\{(0,0)\}$ instead of $A(\xi)$ and $\tilde{\mathbb{Z}}$, respectively.

2. Extension to the case $0\notin A(\xi)$ and $A(\xi)''$ discrete.

We characterized all the numbers ξ for which $A(\xi)'$ is discrete and $0\notin A(\xi)$. Now it is natural to look for the characteristic properties of those numbers ξ for which $0\notin A(\xi)$ and $A(\xi)''$ is discrete. Are these numbers perhaps represented by continued fractions which are pseudo-pseudoperiodic in the sense that in Definition 2 we just replace periodic by pseudoperiodic ?

Indeed, with similar methods as used in section 3 one can show that these continued fractions have the required approximation property, however, there is at least one other class with the same property. A characteristic member of this class is

$$\xi = [2,\underbrace{1,\dots,1}_{\mu},2,2,\underbrace{1,\dots,1}_{\mu},2,2,2,\underbrace{1,\dots,1}_{\mu},\dots\dots,\underbrace{2,\dots,2}_{\mu},\underbrace{1,\dots,1}_{\mu}]_{\mu=1}^{\infty}$$
$$= [2,1,2,1,1,2,2,1,1,2,1,1,1,2,2,1,1,1,2,2,2,1,1,1,2,\dots\dots] .$$

Evidently, this continued fraction is not pseudo-pseudoperiodic; nevertheless, one can show that $A(\xi)''$ is discrete and $0\notin A(\xi)$. Perhaps we shall treat this problem in a subsequent paper.

REFERENCES

1. Baker, A. : Continued fractions of transcendental numbers. Mathematika 9 (1962), 1-8.

2. Ballieu, R. : Sur des suites des nombres liées a une fraction continue régulière. Acad. Roy. Belg. Bull. Cl. Sci. (5) 29 (1943), 165-174.

3. Hall, M. : On the sum and product of continued fractions. Annals of Mathematics (4) 48 (1947), 966-993.

4. Heawood, P.J. : The classification of rational approximation. Proc. London Math. Soc. (2) 20 (1922), 233-250.

5. Jurkat, W.B. and Peyerimhoff, A. : Characteristic approximation properties of quadratic irrationals. Intern. J. of Math. and Math. Sci. (4) 1 (1978), 479-496.

6. Légendre, A.M. : Zahlentheorie. Deutsch von Maser. 2. Auflage. Leipzig 1892.

7. Lekkerkerker, C.G. : Una questione approssimazione diofantea una propieta caratteristica dei numeri quadratici I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 179-185, 257-262.

8. Maillet, E. : Introduction à la théorie des nombres transcendants. Paris 1906.

9. Perron, O. : Die Lehre von den Kettenbrüchen I. Teubner Verlag. Stuttgart 1954.

10. Riederle, M. : Characteristic approximation properties of complex quadratic irrationals. In preparation.

11. Riederle, M. : Short proofs of theorems of Lekkerkerker and Ballieu. To be published in the Intern. J. of Math. and Math. Sci.

12. Schecker, H. : Über die Menge der Zahlen die als Minima quadratischer Formen auftreten. Journal of Number Theory (1) 9 (1977), 121-141.

13. Schmidt, W. : Simultaneous approximation to algebraic numbers by rationals. Acta Math. 125 (1970), 189-201.