J. Math. Biology (1982) 15: 319–338

Journal of
Mathematical
Biology



# Global Asymptotic Stability of a Periodic Solution to an Epidemic Model*

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**Abstract.** In this paper a periodic delay differential equation with spatial spread is investigated. This equation can be used to model the growth of malaria which is transmitted by a mosquito. Using monotone techniques, it is shown that the following bifurcation holds: either the disease dies out or the density of infectious people tends to a spatially homogeneous, time periodic and positive solution.

**Key words:** Epidemic model – Asymptotic stability – Monotone techniques

## 1. Introduction

The purpose of this paper is to investigate the asymptotic behaviour of the problem

$$
\text{(P)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = (1 - u(t,x)) \displaystyle\int_0^\infty \int_\Omega b(t,s,x,y) u(t-s,y)\, dy\, d\eta(s) \\ \qquad\qquad - c(t)u(t,x) + \gamma(t)\, \Delta u(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(s,x) = \phi(s,x), & (s,x) \in \mathbb{R}_- \times \Omega, \\ \dfrac{\partial u}{\partial v}(t,z) = 0, & (t,z) \in \mathbb{R}_+ \times \partial\Omega, \end{cases}
$$

where $\Omega$ is a bounded region in $\mathbb{R}^N$ with $\partial\Omega \in C^2$, $\Delta$ denotes the Laplace operator, $\partial/\partial v$ stands for the outward normal derivative, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

Marcati and Pozio [7] considered problem (P) under the assumptions that the functions $b$, $c$ do not depend on $s$, $t$ and the measure $d\eta$ has compact support in $\mathbb{R}_+$. They used sequences of contracting convex sets in order to prove the global asymptotic stability of a constant solution. Busenberg and Cooke [4] considered the equation without diffusion and the measure $d\eta$ concentrated at $T > 0$. They proved the existence of periodic solutions by means of a fixed point theorem for operators on cones. Moreover, using Lyapunov functions, they showed that these solutions are locally asymptotically stable.

* Research partially supported by NSF Grant MCS 810-4837

0303–6812/82/0015/0319/\$04.00

In this paper, both of the results will be generalized. It is shown that there exists a time periodic solution which is globally asymptotically stable. Both, the existence and the stability property of the periodic solution are simultaneously established by using monotone techniques.

With the equation (P) we want to model the proportion of infectious persons of a communicable disease carried by a mosquito.

The human population is divided into two classes, susceptible and infectious persons, whereas the mosquito population is divided into three classes, infectious, exposed and susceptible mosquitos.

The transmission of the disease is as follows:

a) Susceptible persons can receive the infection only by contact with infectious mosquitos, and susceptible mosquitos can receive the infection only from infectious persons.

b) A susceptible mosquito becomes exposed when it receives the infection from an infected human. It remains exposed for some time and then becomes infectious.

Hence the infection is of S-I-S type in humans and of S-E-I-S type in mosquitos.

We make the following assumptions about the model:

c) The infection in humans does not result in immunity, death or isolation.

d) Both, the mosquito and human population are homogeneously distributed over $\Omega$.

e) The total human population is constant, whereas the total mosquito population is allowed to have seasonal fluctuation.

f) All populations and subpopulations are allowed to diffuse inside $\Omega$, however, migration through $\partial\Omega$ is not allowed.

Denote by

$$u(t,x) = \text{normalized spatial density of infectious persons at time } t \text{ in } x,$$

$$v(t,x) = \text{normalized spatial density of susceptible persons at time } t \text{ in } x,$$

where the normalization is done with respect to the spatial density of the total population. Hence, the following equation holds

$$u(t,x) + v(t,x) = 1, \qquad (t,x) \in \mathbb{R}_+ \times \Omega. \tag{1.1}$$

Moreover, define

$$I(t,x) = \text{spatial density of infectious mosquitos,}$$

$$E(t,x) = \text{spatial density of mosquitos which become exposed at time } t \text{ in } x.$$

If $\alpha(t)$ denotes the contact rate of humans and mosquitos, the density of new infections in humans per unit time is given by

$$\alpha(t)v(t,x)I(t,x) = \alpha(t)(1 - u(t,x))I(t,x).$$

The density of vanishing infections per unit time is given by

$$c(t)u(t,x),$$

where $c(t)$ stands for the cure and death rate of infected humans. Moreover, the difference of the densities of arriving and leaving infections per unit time is given by

$$\gamma(t)\,\Delta u(t,x).$$

When combining this information we obtain the following equation

$$\frac{\partial u}{\partial t}(t,x) = \gamma(t)\,\Delta u(t,x) - c(t)u(t,x) + (1-u(t,x))\alpha(t)I(t,x). \tag{1.2}$$

If the mosquito population is large enough, we can assume that

$$E(t,x) = h(t)u(t,x), \tag{1.3}$$

where $h(t)$ is a positive function.

If we denote by $G(t,s,x,y)$ the proportion of vectors which arrive in $x$ at time $t$, starting from $y$ at time $t-s$, then

$$\int_\Omega G(t,s,x,y)\,dy = 1 \qquad \text{and} \qquad \int_\Omega G(t,s,x,y)E(t-s,y)\,dy$$

is the density of vectors which became exposed at time $t-s$ and are in $x$ at time $t$. Let $d\eta_0(s)$, $s \geqslant 0$, denote the proportion of vectors which are still infectious $s$ units of time after they became exposed, then

$$\begin{aligned} I(t,x) &= \int_0^\infty \int_\Omega G(t,s,x,y)E(t-s,y)\,dy\,d\eta_0(s) \\ &= \int_0^\infty \int_\Omega G(t,s,x,y)h(t-s)u(t-s,y)\,dy\,d\eta_0(s). \end{aligned} \tag{1.4}$$

When combining the Eqs. (1.2) and (1.4) we obtain the differential equation in (P), where

$$b(t,s,x,y) = \alpha(t)h(t-s)G(t,s,x,y)\int_0^\infty d\eta_0(r) \quad \text{and} \quad \eta(s) = \eta_0(s)\left(\int_0^\infty d\eta_0(r)\right)^{-1}.$$

Moreover, it follows that

$$a(t,s) := \int_\Omega b(t,s,x,y)\,dy$$

does not depend on $x \in \Omega$.

Furthermore, the following hypothesis is included:

g) The functions $c(t)$, $a(t,\cdot)$ have seasonal repetition, i.e. they are periodic in the variable $t$ with period $\omega > 0$.

Finally, the condition f) that the human population is confined within $\Omega$ is expressed by

$$\frac{\partial u}{\partial \nu}(t,z) = 0, \qquad (t,z) \in \mathbb{R}_+ \times \partial\Omega. \tag{1.5}$$

In the case that the density of infectious people $u(t,x)$ oscillates very rapidly, the Eq. (1.3) might not be a good approximation of the real situation any more. Also, one would like to drop the assumption d). Maybe in a subsequent paper we can relax some of the assumptions and establish a result analogous to the one in this paper.

## 2. Statement of the Problem

We want to prove the existence of a solution to the problem (P) under the following assumptions:

($A_1$) The set $\Omega$ is a bounded region in $\mathbb{R}^N$ with $\partial\Omega \in C^2$.
($A_2$) The function $\gamma(t)$ is nonnegative and continuous.
($A_3$) The measure $d\eta$ is nonnegative on $\mathbb{R}_+$ and $\int_0^\infty d\eta(s) = 1$.
($A_4$) The inequalities $b(t,s,x,y) > 0$ and $c(t,s) > 0$ hold for all $t \in \mathbb{R}$, $s \in \mathbb{R}_+$ and $x, y \in \Omega$. Moreover, the functions $b$, $c$ are bounded and continuous.
($A_5$) The function

$$a(t,s) := \int_\Omega b(t,s,x,y)\,dy$$

is independent of $x \in \Omega$, and both, $a(t,s)$ and $c(t)$ are periodic in the variable $t$ with period $\omega > 0$.
($A_6$) The initial function $\phi(s,x)$ is continuous and satisfies $0 \leqslant \phi(s,y) \leqslant 1$, $(s,y) \in \mathbb{R}_- \times \Omega$.

For any function $u(s,x)$ defined on $(-\infty, \tau] \times \Omega$ denote by $u_\tau$ the past history of $u$ at $\tau$, i.e. $u_\tau(s,x) = u(s+\tau, x)$, $(s,x) \in \mathbb{R}_- \times \Omega$. If in addition the function $u$ is bounded and continuous, define

$$(J(t,u_\tau))(x) = \int_0^\infty \int_\Omega b(t,s,x,y)u(\tau - s, y)\,dy\,d\eta(s).$$

Consider the Banach space $E = C(\bar{\Omega}, \mathbb{R})$ endowed with the norm

$$|v|_E = \max\{|v(x)|: x \in \bar{\Omega}\}, \qquad v \in E,$$

and the operator $A$ with domain

$$D(A) = \left\{u \in E: \Delta u \in E, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}$$

and defined by

$$(Au)(x) = \Delta u(x), \qquad u \in D(A).$$

With the substitution $(u(t))(x) = u(t,x)$ we can formulate problem (P) as an abstract problem in the Banach space $E$

$$(\mathrm{P}_1) \qquad \begin{cases} \dfrac{du}{dt} = \gamma(t)Au + (1 - u(t))J(t,u_t) - c(t)u(t), \\ u_0 = \phi. \end{cases}$$

## 3. Existence and Uniqueness of Solutions

The hypothesis that $\partial\Omega \in C^2$ guarantees that for every $w \in E$, $\lambda > 0$ there exists a $u \in D(A)$ which solves the equation $Au - \lambda u = w$, (e.g. see [6]). Moreover, from [8, Th. 13, p. 78] it follows that the solution $u$ is unique and $|u|_E \leqslant (1/\lambda)|w|_E$. Hence we can apply the Hille-Yosida theorem (e.g. see [2, Th. 4.2.1, p. 171]) which establishes the existence of a strongly continuous semigroup of operators, denoted by $e^{tA} \in C(E, E)$, with infinitesimal generator $A$ and

$$\|e^{tA}\| \leqslant 1, \qquad t \in \mathbb{R}_+, \tag{3.1}$$

$$e^{tA}p = p, \qquad p \text{ is a constant function over } \Omega. \tag{3.2}$$

Suppose that $v \in E$, $0 \leqslant v$ and define $p = |v|_E$. Clearly, we can consider $p$ as a constant function over $\bar{\Omega}$. From (3.1) and (3.2) it follows that

$$p - e^{tA}v = e^{tA}p - e^{tA}v = e^{tA}(p - v) \leqslant |p - v|_E \leqslant p$$

and hence

$$0 \leqslant e^{tA}v, \qquad v \geqslant 0, \qquad t \in \mathbb{R}_+. \tag{3.3}$$

According to the assumption ($A_2$) we can define the operator

$$K(s, t) = \exp\left(\int_s^t \gamma(r)\, dr\, A\right), \qquad s \leqslant t.$$

The following equalities are easily seen to be true:

$$K(t, t) = I, \tag{3.4}$$

$$K(s, \tau)K(\tau, t) = K(s, t), \qquad s \leqslant \tau \leqslant t, \tag{3.5}$$

$$\frac{d}{ds}K(s, t)u = -\gamma(s)AK(s, t)u = -\gamma(s)K(s, t)Au, \qquad u \in D(A). \tag{3.6}$$

Now we can put problem ($P_1$) in its mild form

$$(\mathrm{P}_2) \qquad \begin{cases} u(t) = K(0, t)u(0) + \displaystyle\int_0^t K(s, t)((1 - u(s))J(s, u_s) - c(s)u(s))\, ds, \\ u_0 = \phi. \end{cases}$$

It follows from the contracting mapping principle that there exists a $\delta > 0$ such that problem ($P_2$) has a unique solution $u \in C((-\infty, \delta], E)$. Furthermore, the solution $u$ exists globally if and only if it is bounded on any finite interval.

In order to show that the solution $u$ is bounded we need the following lemma.

**Lemma 1.** *Suppose that the function u solves the integral equation*

$$u(t) = K(0, t)u(0) + \int_0^t K(s, t)f(s, u_s)\, ds, \qquad t \in [0, \delta),$$

*where $f$ is a continuous function. If $h$ is a real valued and continuous function on $[0, \delta)$, then the following identity holds for all $t \in [0, \delta)$:*

$$u(t) = \exp\left(-\int_0^t h(r)\, dr\right)K(0, t)u(0)$$
$$+ \int_0^t \exp\left(-\int_s^t h(r)\, dr\right)K(s, t)(h(s)u(s) + f(s, u_s))\, ds.$$

*Proof.* For $0 \leqslant s,\ t < \delta$ define

$$k(s,t) = \exp\left(-\int_s^t h(r)\,dr\right), \qquad \text{then} \qquad \frac{d}{ds}k(s,t) = h(s)k(s,t).$$

Consider the term

$$\begin{aligned}
&\int_0^t h(s)k(s,t)K(s,t)u(s)\,ds \\
&\quad = \int_0^t h(s)k(s,t)K(s,t)\left(K(0,s)u(0) + \int_0^s K(\tau,s)f(\tau,u_\tau)\,d\tau\right)ds \\
&\quad = \int_0^t h(s)k(s,t)K(0,t)u(0)\,ds + \int_0^t h(s)k(s,t)\int_0^s K(\tau,t)f(\tau,u_\tau)\,d\tau\,ds.
\end{aligned}$$

Observe that the function $K(\tau,t)f(\tau,u_\tau)$ is continuous in $\tau$ and that

$$\int_0^t h(s)k(s,t)\,ds = 1 - k(0,t).$$

Integration by parts leads to the identity

$$\begin{aligned}
\int_0^t h(s)k(s,t)K(s,t)u(s)\,ds &= K(0,t)u(0) - k(0,t)K(0,t)u(0) \\
&\quad + \left[k(s,t)\int_0^s K(\tau,t)f(\tau,u_\tau)\,d\tau\right]_{s=0}^t \\
&\quad - \int_0^t k(s,t)K(s,t)f(s,u_s)\,ds \\
&= -\,k(0,t)K(0,t)u(0) - \int_0^t k(s,t)K(s,t)f(s,u_s)\,ds \\
&\quad + K(0,t)u(0) + \int_0^t K(\tau,t)f(\tau,u_\tau)\,d\tau. \qquad (3.7)
\end{aligned}$$

The assumption of the lemma and the equation (3.7) imply

$$u(t) = k(0,t)K(0,t)u(0) + \int_0^t k(s,t)K(s,t)(h(s)u(s) + f(s,u_s))\,ds$$

and the proof of the lemma is finished. □

## 4. Invariance of Solutions

With the equation ($P_2$) we wish to describe the density of a certain population. Therefore, we are primarily interested in solutions $u(t)$ which satisfy $0 \leqslant u(t) \leqslant 1$.

In order that our model makes sense, any mild solution $u(t)$ of ($P_2$) with $0 \leqslant u_0 \leqslant 1$ must satisfy $0 \leqslant u(t) \leqslant 1$ for all values $t$ where $u$ is defined. If this is true then the solution $u$ exists globally. In this section even more will be shown, namely if

$u$, $v$ are mild solutions of $(P_2)$ such that $0 \leqslant u_0 \leqslant v_0 \leqslant 1$, then

$$0 \leqslant u_t \leqslant v_t \leqslant 1, \qquad t \in \mathbb{R},$$

a result which is basic in order to prove the existence and stability properties of periodic solutions.

Define the one dimensional subspace

$$F = \{u \in E: u \text{ is a constant function over } \bar{\Omega}\},$$

and the sets

$$U = \{u_0 \in C(\mathbb{R}_-, E): 0 \leqslant u_0 \leqslant p_0\},$$
$$V = U \cap C(\mathbb{R}_-, F).$$

According to the contracting mapping principle there exists a $\delta > 0$ such that the problem

$$(\mathrm{P}_2') \quad \begin{cases} u(t) = u(0) + \displaystyle\int_0^t (-c(s)u(s) + (1 - u(s)) \int_0^\infty a(s,\tau)u(s-\tau)\,d\eta(\tau))\,ds, \\ u_0 = \phi \in v \end{cases}$$

has a unique solution $u \in C((-\infty, \delta), F)$.

Since the integrand in $(\mathrm{P}_2')$ is continuous we can differentiate and obtain

$$(\mathrm{P}_1') \qquad \begin{cases} \dfrac{du}{dt}(t) = -c(t)u(t) + (1 - u(t)) \displaystyle\int_0^\infty a(s,\tau)u(t-\tau)\,d\eta(\tau), \\ u_0 = \phi \in v. \end{cases}$$

Since the operator $A$ is identically zero on $F$ and because of $(\mathrm{A}_5)$ for any $\phi \in V$ the problems $(\mathrm{P}_1)$ and $(\mathrm{P}_2)$ coincide with the problems $(\mathrm{P}_1')$ and $(\mathrm{P}_2')$, respectively.

Let $u(t)$ be a solution of $(\mathrm{P}_1')$ on $[0, \delta)$ with $u(0) \leqslant 1$. Moreover, define $\xi = \sup\{t \geqslant 0: u(t) \leqslant 1\}$ and suppose that $\xi < \delta$. Since the function $u$ is continuous we have that $u(\xi) = 1$ and

$$\frac{du}{dt}(\xi) = -c(\xi) < 0. \tag{4.1}$$

The right hand side of the equation in $(\mathrm{P}_1')$ is continuous and so is $u'$. Therefore, the following identity is true

$$u(\xi + s) = 1 + \int_\xi^{\xi+s} u'(\tau)\,d\tau, \qquad s \in (-\xi, \delta - \xi). \tag{4.2}$$

From (4.1), (4.2) and the continuity of $u'$ it follows that there exists an $\varepsilon > 0$ such that

$$u(\xi + s) \leqslant 1, \qquad 0 \leqslant s \leqslant \varepsilon.$$

This is obviously in contradiction to the choice of $\xi$. Therefore, $\xi = \delta$ and $u(t) \leqslant 1$ for $t \in [0, \delta)$. Moreover, from (4.2) it is easily seen that $u(t) < 1$, $0 < t < \delta$.

Now we are in the position to prove the main result of this section.

**Theorem 1.** *For each $\phi \in U$, the initial value problem* $(P_2)$ *has a unique solution $u(t)$ on all of $\mathbb{R}_+$ and $u_t \in U$, $t \in \mathbb{R}$. Moreover, if $v(t)$ is another solution of* $(P_2)$ *with $v_0 \in U$ and $u_0 \leqslant v_0$ then $u_t \leqslant v_t$, $t \in \mathbb{R}$.*

*Proof.* i) Let $u(t)$, $v(t)$ both be solutions of $(P_2)$ for $t \in [0, \delta]$, $\delta > 0$. Furthermore, assume that $u_0$, $v_0 \in U$ and $u_0 \leqslant v_0$. Define the auxiliary function

$$w(t) = v(t) - u(t).$$

When subtracting the integral equations for $u$ and $v$, we obtain

$$w(t) = K(0, t)w(0) + \int_0^t K(s, t)g(s, w_s)\, ds,$$

$$w_0 = v_0 - u_0 \geqslant 0, \tag{4.4}$$

where $g(s, w_s) = (-c(s) - J(s, u_s))w(s) + (1 - v(s))J(s, w_s)$. Define the number

$$\alpha = \sup_{0 \leqslant s \leqslant \delta} |c(s) + J(s, u_s)|_E$$

and the function

$$\beta(s) = \alpha - c(s) - J(s, u_s) \geqslant 0.$$

According to Lemma 1, the function $w(t)$ also fulfills the equation

$$w(t) = e^{-\alpha t}K(0, t)w(0) + \int_0^t e^{-\alpha(t-s)}K(s, t)((1 - v(s))J(s, w_s) + \beta(s)w(s))\, ds. \tag{4.5}$$

ii) For the moment we assume in addition that $v_0 = 1$. By the remarks at the beginning of this section we have

$$1 - v(s) \geqslant 0, \qquad s \in [0, \delta]. \tag{4.6}$$

Define

$$p(t) = \sup\{q \in F\colon q \leqslant w(s),\ s \leqslant t\}, \qquad t \leqslant \delta,$$

and

$$\xi = \sup\{t \leqslant \delta\colon 0 \leqslant p(t)\}.$$

Since $u_0 \leqslant v_0$ we have that $\xi \geqslant 0$.

Suppose by contradiction that $\xi < \delta$. Without loss of generality we can assume that $\xi = 0$. Therefore, and since the function $p$ is nonincreasing we have $p(t) < 0$, $t \in (0, \delta]$.

The following inequalities are true

$$\beta(s)w(s) \geqslant \beta(s)p(s) \tag{4.7}$$

and

$$\begin{aligned} J(s, w_s) &= \int_0^\infty \int_\Omega b(S, T, \cdot, y)w(s - \tau, y)\, dy\, d\eta(\tau) \\ &\geqslant \int_0^\infty \int_\Omega b(S, T, \cdot, y)p(s - \tau)\, dy\, d\eta(\tau) \end{aligned}$$

$$\geqslant p(s) \int_0^\infty a(s,\tau)\,d\eta(\tau). \tag{4.8}$$

Let

$$\Theta = \sup\left\{|1 - v(s)|_E \int_0^\infty a(s,\tau)\,d\eta(\tau) + \beta(s) \colon 0 \leqslant s \leqslant \delta\right\}.$$

Since $w(0) \geqslant 0$ and because of (4.6), (4.7) it follows from the equation (4.5) that

$$\begin{aligned} w(t) &\geqslant \int_0^t e^{-\alpha(t-s)} p(s)\left(|1 - v(s)|_E \int_0^\infty a(s,\tau)\,d\eta(\tau) + \beta(s)\right) ds \\ &\geqslant t\Theta p(t), \qquad t \in [0,\delta]. \end{aligned} \tag{4.9}$$

According to the definition of $p(t)$ we must have

$$\begin{aligned} p(t) &= \sup\{q \in F\colon q \leqslant w(s),\ 0 \leqslant s \leqslant t\} \\ &\geqslant \sup\{q \in F\colon q \leqslant s\Theta p(s),\ 0 \leqslant s \leqslant t\} \\ &\geqslant t\Theta p(t), \qquad t \in [0,\delta]. \end{aligned} \tag{4.10}$$

If we choose $t > 0$ such that $t\Theta < 1$ we get a contradiction to the inequality (4.10), because $p(t) < 0$ and $\Theta > 0$.

Therefore, $p(t) \geqslant 0$ for $t \leqslant \delta$ and

$$v_\delta - u_\delta = w_\delta \geqslant 0.$$

We observe that $u = 0$ is the solution of ($P_2$) with $\phi = 0$. This implies that

$$0 \leqslant v_\delta \leqslant 1.$$

Hence the solution $v$ exists globally and $v_\delta \in U$ for all $\delta \geqslant 0$. Furthermore, any solution $u(t)$ of ($P_2$) which exists on $[0,\delta]$ satisfies

$$u_\delta \leqslant 1, \qquad u_0 \in U. \tag{4.11}$$

iii) Now we drop the assumption on $v_0$ and assume only that $v_0 \in U$. We have just shown that (4.6) holds in this case, too, and by the same procedure as in ii) we obtain that

$$u_\delta \leqslant v_\delta, \qquad \text{if} \qquad u_0 \leqslant v_0.$$

In particular if $u = 0$ we have $0 \leqslant v_\delta$. This combined with (4.11) leads to

$$0 \leqslant v_\delta \leqslant 1$$

and the proof of the theorem is finished. □

As a consequence of Theorem 1 and the remarks at the beginning of this section we obtain the following result.

**Corollary 1.** *If $u(t)$ is a solution of ($P_2$) with $u_0 \in U$ then*

$$|u(t)|_E < 1, \qquad t > 0.$$

## 5. Stability Properties of Periodic Solutions

For the rest of this paper we shall always choose the initial function $\phi$ in $U$. The main result will be proved in this section: if the zero solution is the only $\omega$-periodic solution of $(P_2)$ then all solutions tend to zero. In the other case there exists a unique $\omega$-periodic solution $\hat{u}$ which attracts all other solutions except the zero solution.

Define

$$\sigma = \sup\left\{s \geqslant 0: \int_s^\infty d\eta(\tau) > 0\right\}, \qquad 0 \leqslant \sigma \leqslant \infty,$$

and

$$\chi(u_0) = \sup\{|u_0(\tau)|_E: -\sigma \leqslant \tau \leqslant 0\}, \qquad u_0 \in U.$$

If $\chi(u_0) = 0$ then $J(s, u_0) = 0$ and the function $u(t) = 0$, $t \geqslant 0$, solves the equation $(P_2)$.

The following lemma gives a lower bound for a solution $u(t)$ with $\chi(u_0) > 0$.

**Lemma 2.** *Suppose that the function $u(t)$ solves the integral equation* $(P_2)$ *and* $\chi(u_0) > 0$. *Then there exist constants $t_0, p > 0$ such that*

$$u(t) \geqslant p \exp\left(-\int_{t_0}^t c(r)\, dr\right), \qquad t \geqslant t_0.$$

*Proof.* Our assumption ensures the existence of a $\xi \in \mathbb{R}_+ \cap [0, \sigma]$ such that

$$|u(-\xi)|_E > 0. \tag{5.1}$$

From the definition of $\sigma$ it follows that

$$\int_{\max(0,\xi-\varepsilon)}^\infty d\eta(s) > 0 \qquad \text{for all} \qquad \varepsilon > 0.$$

Now it is not hard to see that there must be a $\mu \in \mathbb{R}_+$, $\mu \geqslant \xi$, such that

$$\int_{\max(0,\mu-\varepsilon)}^{\mu+\varepsilon} d\eta(s) > 0 \qquad \text{for all} \qquad \varepsilon > 0. \tag{5.2}$$

If we define $\theta = \mu - \xi \geqslant 0$ it follows from $(A_4)$ and (5.1) that

$$\int_\Omega b(\theta, \mu, x, y) u(-\xi, y)\, dy > 0 \qquad \text{for all} \qquad x \in \bar{\Omega}.$$

Since the integral above is continuous in the variables $\mu$, $\xi$ there must exist $q, \delta > 0$ such that

$$\int_\Omega b(\theta, s, x, y) u(-\xi + \mu - s, y)\, dy \geqslant q \tag{5.3}$$

for all $x \in \bar{\Omega}$ and $\max(0, \mu - \delta) \leqslant s \leqslant \mu + \delta$. Now

$$J(\theta, u_\theta) = \int_0^\infty \int_\Omega b(\theta, s, \cdot, y) u(\theta - s, y)\, dy\, d\eta(s)$$

$$\geqslant \int_{\max(0,\mu-\delta)}^{\mu+\delta} \int_{\Omega} b(\theta, s, \cdot, y) u(-\xi + \mu - s, y)\, dy\, d\eta(s)$$

$$\geqslant \int_{\max(0,\mu-\delta)}^{\mu+\delta} q\, d\eta(s)$$

and because of (5.2) there is a $q_0 > 0$ such that

$$J(\theta, u_\theta) \geqslant q_0.$$

Since the function $J(\theta, u_\theta)$ is continuous in $\theta$ we can assume without loss of generality that for some $\lambda > 0$ the inequality

$$J(s, u_s) \geqslant q_0 \tag{5.4}$$

holds for all $s \in [\theta, \theta + \lambda]$.

From (5.4) and Corollary 1 it follows that

$$(1 - u(s)) J(s, u_s) \geqslant q_1(s), \qquad s \in (\theta, \theta + \lambda], \tag{5.5}$$

where the function $q_1$ is positive and $q_1(s) \in F$, $s \in (\theta, \theta + \lambda]$. For any $\tau \geqslant 0$, according to Lemma 1 the function $u$ also satisfies the equation

$$u(t) = \exp\left(-\int_\tau^t c(r)\, dr\right) K(\tau, t) u(\tau)$$
$$+ \int_\tau^t \exp\left(-\int_s^t c(r)\, dr\right) K(s, t)(1 - u(s)) J(s, u_s)\, ds, \qquad t \geqslant \tau. \tag{5.6}$$

In particular, if we choose $\tau = \theta$ and $t = \theta + \lambda$ it follows from (5.5) and (5.6) that

$$u(\theta + \lambda) \geqslant \int_\theta^{\theta+\lambda} \exp\left(-\int_s^{\theta+\lambda} c(r)\, dr\right) q_1(s)\, ds =: p > 0.$$

If we choose $\tau = \theta + \lambda$ in (5.6) we obtain that

$$u(t) \geqslant \exp\left(-\int_\tau^t c(r)\, dr\right) K(\tau, t) u(\tau) \geqslant \exp\left(-\int_\tau^t c(r)\, dr\right) p, \qquad t \geqslant \tau,$$

and the proof of the lemma is finished. □

So far we have not made use of the assumption ($A_5$) that the functions $a$, $c$ are periodic. However, this will be done in the next theorem which contains the basic result of this paper.

**Theorem 2.** *There exists an $\omega$-periodic solution $\hat{u}(t)$ of* ($P'_1$) *which is globally asymptotically stable in $W = \{u_0 \in U : \chi(u_0) > 0\}$. Either $\hat{u} \equiv 0$ or $0 < \hat{u}(t) < 1$ for all $t \in \mathbb{R}$.*

*Proof.* i) If $v(t)$ denotes the solution of ($P'_1$) with $v_0 = 1$ then it follows from Corollary 1 that $v_\omega \leqslant 1 = v_0$. Hence, if $w(t)$ denotes the solution of ($P'_1$) with $w_0 = v_\omega$ then $w(t) \leqslant v(t)$, $t \in \mathbb{R}$ according to Theorem 1. From the assumption ($A_5$) we conclude that $w(t) = v(t + \omega)$ and

$$v(t + \omega) \leqslant v(t), \qquad t \in \mathbb{R}. \tag{5.7}$$

Hence, for fixed $t$ the sequence $v(k\omega + t)$ is monotone in $k \in \mathbb{N}$ and we can define

$$\hat{u}(t) = \lim_{k \to \infty} v(k\omega + t), \qquad t \in \mathbb{R}. \tag{5.8}$$

From the identity

$$\hat{u}(t) = \lim_{k \to \infty} v(k\omega + t) = \lim_{k \to \infty} v((k-1)\omega + t + \omega) = \hat{u}(t + \omega) \tag{5.9}$$

we see that the function $\hat{u}$ is $\omega$-periodic.

A straightforward argument shows that every solution $u(t)$ of $(P'_1)$ with $u_0 \in U$ satisfies the inequality

$$|u'(t)| \leqslant c(t) + \int_0^\infty a(t,s)\,d\eta(s)$$

and hence, the solutions of $(P'_1)$ with $u_0 \in U$ are equicontinuous on $\mathbb{R}_+$. In particular, we obtain that the convergence in (5.8) is uniform on bounded intervals and the function $\hat{u}$ is continuous.

Given $\varepsilon > 0$, there exist $k_0$, $\mu > 0$ such that

$$\int_\mu^\infty d\eta(s) < \varepsilon \quad \text{and} \quad |v(k\omega + s) - \hat{u}(s)| < \varepsilon \quad \text{for} \quad k \geqslant k_0, \quad s \in [-\mu, \omega].$$

Now if $t \in [0, \omega]$ we have

$$\begin{aligned} &|J(t, v_{k\omega+t}) - J(t, \hat{u}_t)| \\ &\quad \leqslant \int_0^\mu a(t,s)|v(k\omega + t - s) - \hat{u}(t-s)|\,d\eta(s) + \int_\mu^\infty a(t,s)\,d\eta(s) \\ &\quad \leqslant 2\varepsilon|a|_{\mathbb{R}^2}, \end{aligned}$$

which tells us that $J(t, v_{k\omega+t})$ converges to $J(t, \hat{u}_t)$ uniformly on $[0, \omega]$.

Moreover, the following equality holds for all $t \geqslant 0$:

$$v(k\omega + t) = v(k\omega) + \int_0^t ((1 - v(k\omega + s))J(s, v_{k\omega+s}) - c(s)v(k\omega + s))\,ds. \tag{5.10}$$

The integrand in (5.10) converges uniformly on $[0, \omega]$ as $k$ tends to infinity and hence

$$\hat{u}(t) = \hat{u}(0) + \int_0^t ((1 - \hat{u}(s))J(s, \hat{u}_s) - c(s)\hat{u}(s))\,ds, \qquad t \in [0, \omega].$$

We have just proved that $\hat{u}$ is a $\omega$-periodic solution of $(P'_1)$ and $(P'_2)$. Moreover, since the convergence in (5.8) is uniform on bounded intervals we have

$$\lim_{k \to \infty} \sup_{0 \leqslant t \leqslant \omega} |v(k\omega + t) - \hat{u}(k\omega + t)| = 0$$

and hence

$$\lim_{s \to \infty} |v(s) - \hat{u}(s)| = 0. \tag{5.11}$$

Furthermore, if $u(t)$ is any solution of $(P_2)$ with $u_0 \in U$, then it follows from Theorem 1 that

$$u(t) \leqslant v(t), \qquad t \in \mathbb{R}. \tag{5.12}$$

ii) We shall now prove the stability asserted in the theorem. If $\hat{u} \equiv 0$ then the result follows by an argument combining (5.11) and (5.12).

This allows us to restrict ourselves to the case that $\hat{u} \not\equiv 0$. But then there exists a $\tau \in \mathbb{R}$ such that $\hat{u}(\tau) > 0$ and from the equation (5.6) it follows that

$$\hat{u}(t) \geqslant \exp\left( - \int_\tau^t c(r)\, dr \right) \hat{u}(\tau), \qquad t \geqslant \tau.$$

Now from Corollary 1 and the periodicity of the function $\hat{u}$ we obtain that

$$0 < \hat{u}(t) < 1, \qquad t \in \mathbb{R}.$$

Consider any solution $w(t)$ of $(P'_2)$ with $\chi(w_0) > 0$, $w_0 \in V$. Since $u, w \in C(\mathbb{R}, \mathbb{R})$ we can define

$$\delta = \liminf_{t \to \infty} \frac{w(t)}{\hat{u}(t)}. \tag{5.13}$$

Suppose by contradiction that $0 < \delta < 1$. There exists a $q$, $0 < q < 1$, such that

$$q^3 > \frac{1 - \hat{u}(t)}{1 - \delta \hat{u}(t)}, \qquad t \in \mathbb{R}. \tag{5.14}$$

Moreover, since the functions $\hat{u}(t)$ and $a(t, \cdot)$ are periodic, we can find a $\lambda > 0$ such that

$$qJ(t, \hat{u}_t) \leqslant \int_0^\lambda a(t, s) \hat{u}(t - s)\, d\eta(s), \qquad t \in \mathbb{R}. \tag{5.15}$$

From the definition of $\delta$ it follows that there exists $\tau > 0$ such that

$$w(t) \geqslant q\delta \hat{u}(t), \qquad t \geqslant \tau. \tag{5.16}$$

For any $t \geqslant \tau + \lambda$ we have that

$$\begin{aligned} J(t, w_t) &\geqslant \int_0^\lambda a(t, s) w(t - s)\, d\eta(s) \\ &\geqslant \int_0^\lambda a(t, s) q\delta \hat{u}(t - s)\, d\eta(s) \\ &\geqslant q^2 \delta J(t, \hat{u}_t). \end{aligned} \tag{5.17}$$

Moreover, by the definition of $\delta$ there exists a sequence $\xi_k$, $\lim_{k \to \infty} \xi_k = \infty$, such that

$$\lim_{k \to \infty} (w(\xi_k) - \delta \hat{u}(\xi_k)) = 0. \tag{5.18}$$

Without loss of generality we can assume that $\xi_k \geqslant \tau + \lambda$ and

$$(1 - w(\xi_k)) \geqslant q(1 - \delta \hat{u}(\xi_k)), \qquad k \in \mathbb{N}. \tag{5.19}$$

Hence, if we write $\xi = \xi_k$ we obtain

$$(1 - w(\xi))J(\xi, w_\xi) \geqslant q^3\delta(1 - \delta\hat{u}(\xi))J(\xi, \hat{u}_\xi) \tag{5.20}$$

and

$$\begin{aligned} w'(\xi) - \delta\hat{u}'(\xi) &= (1 - w(\xi))J(\xi, w_\xi) - \delta(1 - \hat{u}(\xi))J(\xi, \hat{u}_\xi) - c(\xi)(w(\xi) - \delta\hat{u}(\xi)) \\ &\geqslant \delta J(\xi, \hat{u}_\xi)(q^3(1 - \delta\hat{u}(\xi)) - (1 - \hat{u}(\xi))) - c(\xi)(w(\xi) - \delta\hat{u}(\xi)). \quad (5.21) \end{aligned}$$

Using (5.14) and (5.18) it follows from (5.21) that there exist $k_0$, $\varepsilon > 0$ such that

$$w'(\xi_k) - \delta\hat{u}'(\xi_k) \geqslant \varepsilon, \qquad k \geqslant k_0. \tag{5.22}$$

Any solution $u(t)$ of $(\mathrm{P}_1')$ is uniformly continuous on $\mathbb{R}_+$ and since the functions $a(t, \cdot)$, $c(t)$ are bounded, continuous and periodic, it follows from the equation $(\mathrm{P}_1')$ that the function $u'(t)$ is uniformly continuous on $\mathbb{R}_+$.

This implies that we can find a $\zeta > 0$ such that

$$w'(\xi_k + s) - \delta\hat{u}'(\xi_k + s) \geqslant \frac{\varepsilon}{2}, \qquad k \geqslant k_0, \qquad s \in [-\zeta, \zeta].$$

By the mean value theorem we obtain

$$w(\xi_k - \zeta) - \delta\hat{u}(\xi_k - \zeta) \leqslant w(\xi_k) - \delta\hat{u}(\xi_k) - \zeta\frac{\varepsilon}{2}$$

and therefore

$$\liminf_{k \to \infty} \frac{w(\xi_k - \zeta)}{\hat{u}(\xi_k - \zeta)} - \delta \leqslant \liminf_{k \to \infty} \frac{-\zeta\varepsilon}{2\hat{u}(\xi_k - \zeta)} < 0$$

which is in contradiction to the choice of $\delta$.

Suppose now that $\delta = 0$. Choose $q$, $\lambda$ as in (5.14), (5.15). Since $\chi(w_0) > 0$, according to Lemma 2, we can assume without loss of generality that for some $\delta_0 > 0$

$$w(t) \geqslant \delta_0\hat{u}(t), \qquad t \in [0, \lambda]. \tag{5.23}$$

Define

$$\xi = \sup\{t \geqslant 0 \colon w(r) \geqslant \delta_0\hat{u}(r), 0 \leqslant r \leqslant t\}$$

and suppose that $\xi < \infty$. Then

$$w(\xi) = \delta_0\hat{u}(\xi)$$

and similarly as in (5.21) we obtain that

$$w'(\xi) - \delta_0\hat{u}'(\xi) > 0,$$

and we get a contradiction to the choice of $\xi$. Therefore $\delta \geqslant 1$ and (5.13) implies that

$$\liminf_{t \to \infty}(w(t) - \hat{u}(t)) \geqslant 0. \tag{5.24}$$

iii) Given any solution $u(t)$ of $(\mathrm{P}_2)$ with $\chi(u_0) > 0$. Define $w(t)$ to be the solution of $(\mathrm{P}_1')$ such that

$$w(t) = \sup\{p \geqslant 0 \colon p \leqslant u(t)\}, \qquad t \leqslant 0,$$

then according to Lemma 2 we can assume without loss of generality that $\chi(w_0) > 0$.

Let $v(t)$ be as in i) then

$$w(t) \leqslant u(t) \leqslant v(t), \qquad t \in \mathbb{R}, \tag{5.25}$$

and $w(t) - \hat{u}(t) \leqslant u(t) - \hat{u}(t) \leqslant v(t) - \hat{u}(t)$. This leads to the inequality

$$|u(t) - \hat{u}(t)|_E \leqslant |v(t) - \hat{u}(t)| + \max\{0, \ -w(t) + \hat{u}(t)\}$$

and (5.11), (5.24) imply that

$$\lim_{t \to \infty} |u(t) - \hat{u}(t)|_E = 0 \tag{5.26}$$

whenever $\chi(u_0) > 0$.

iv) Now we shall show that the function $\hat{u}$ is also stable. If $\hat{u} = 0$, the result is easily seen to be true according to (5.7), (5.8) and (5.12). In the case that $\hat{u} \neq 0$ let $w(t)$ be a solution of $(\mathrm{P}'_1)$ such that for some $\delta > 0$ the inequality

$$|w(t) - \hat{u}(t)| < \delta \hat{u}(t) \tag{5.27}$$

holds for all $t \leqslant 0$. If for some minimal chosen $\xi \in \mathbb{R}_+$ we would have $w(\xi) = (1 + \delta)\hat{u}(\xi)$ or $w(\xi) = (1 - \delta)\hat{u}(\xi)$, then from the equation in (5.21) we obtain a contradiction to the choice of $\xi$. Hence the inequality (5.27) holds for all $t \in \mathbb{R}$, and by Theorem 1 the solution $\hat{u}$ is stable and the proof of the theorem is finished. □

As a consequence of Theorem 2 we obtain the following result.

**Corollary 2.** *The equations* $(\mathrm{P}_1)$, $(\mathrm{P}_2)$ *have at most one* $\omega$*-periodic solution other than the zero solution.*

**Corollary 3.** *The zero solution is the only nonnegative,* $\omega$*-periodic solution if and only if it is stable.*

*Example.* Suppose that the functions $a$, $c$ are constant. In this case, the function $\hat{u}$ is $\omega$-periodic for every $\omega > 0$ and hence must be a constant. The only constant solution of $(\mathrm{P}'_1)$ other than the zero solution is

$$u = 1 - \frac{c}{a}$$

and therefore

$$\hat{u} = \max\left\{0, 1 - \frac{c}{a}\right\}.$$

## 6. Bifurcation Diagram

We consider the family of differential equations

$$(P(\theta)) \qquad u'(t) = -c(t)u(t) + \theta(1 - u(t))J(t, u_t)$$

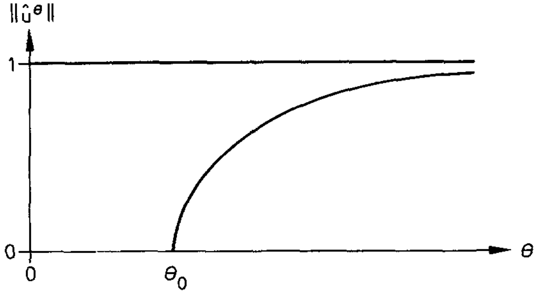


Fig. 1

for $\theta \geqslant 0$. Let $\hat{u}^\theta$ denote the $\omega$-periodic solution of $(P(\theta))$, the existence of which follows from Theorem 2. It will be shown that the function

$$\|\hat{u}^\theta\| := \sup\{\hat{u}^\theta(t) \colon t \in \mathbb{R}\}$$

has the diagram of Fig. 1.

For the proof of this we need the following lemma.

**Lemma 3.** *Suppose that $\hat{u}^\theta \neq 0$ and there exist $\tau$, $\delta > 0$ such that*

$$\tau(1 - \delta\hat{u}^\theta(s)) - \theta(1 - \hat{u}^\theta(s)) > 0, \qquad s \in \mathbb{R}. \tag{6.1}$$

*Then $\hat{u}^\tau(s) \geqslant \delta\hat{u}^\theta(s)$, $s \in \mathbb{R}$.*

*Proof.* Let $v(t)$ be the solution of $(P(\tau))$ with $v_0 = 1$. Define

$$\xi = \sup\{t \colon v(t) > \delta\hat{u}^\theta\}$$

and suppose that $\xi < \infty$. Then

$$\begin{aligned}\frac{d(v - \delta\hat{u}^\theta)}{dt}(\xi) &= \tau(1 - v(\xi))J(\xi, v_\xi) - \theta(1 - \hat{u}^\theta(\xi))J(\xi, \delta\hat{u}^\theta_\xi) \\ &\geqslant J(\xi, \delta\hat{u}^\theta_\xi)(\tau(1 - \delta\hat{u}^\theta(\xi)) - \theta(1 - \hat{u}^\theta(\xi))) > 0.\end{aligned}$$

This is obviously in contradiction to the choice of $\xi$ and hence $v(t) > \delta\hat{u}^\theta(t)$, $t \in \mathbb{R}$. From Theorem 2 it follows that $\hat{u}^\tau(t) \geqslant \delta\hat{u}^\theta(t)$, $t \in \mathbb{R}$, and the proof of the lemma is finished. □

Choose $\theta$ large enough such that

$$q := 1 - \frac{1}{\theta}\left|\frac{c(\cdot)}{J(\cdot, 1)}\right|_{\mathbb{R}} > 0$$

and let $w(t)$ be the solution of $(P(\theta))$ with $w_0 = 1$. Suppose that there is a minimal $\xi \in \mathbb{R}$ with $w(\xi) = q$. Then

$$\begin{aligned}w'(\xi) &= -c(\xi)q + \theta(1 - q)J(\xi, w_\xi) \\ &\geqslant -c(\xi)q + \theta(1 - q)qJ(\xi, 1) > 0\end{aligned}$$

which is impossible. Therefore, $w(t) \geqslant q$, and by Theorem 2 we have

$$\lim_{t \to \infty} |\hat{u}^\theta(t) - w(t)| = 0$$

which implies that

$$\hat{u}^{\theta}(t) \geqslant 1 - \frac{1}{\theta}\left|\frac{c(\cdot)}{J(\cdot,1)}\right|_{\mathbb{R}}, \qquad t \in \mathbb{R}, \qquad \theta > 0. \tag{6.2}$$

As a consequence of (6.2) we obtain that

$$\lim_{\theta \to \infty} \|\hat{u}^{\theta}\| = 1. \tag{6.3}$$

Choose $\tau$, $\theta$ such that $\tau > \theta$ and $\|\hat{u}^{\theta}\| > 0$. We can find a $\delta > 1$ such that the inequality (6.1) holds which implies that

$$\hat{u}^{\tau}(s) > \hat{u}^{\theta}(s), \qquad \tau > \theta, \qquad \|\hat{u}^{\theta}\| > 0, \qquad s \in \mathbb{R}. \tag{6.4}$$

Define

$$\theta_0 = \inf\{\theta\colon \|\hat{u}^{\theta}\| > 0\},$$

then an easy consequence of (6.4) gives us

$$\hat{u}^{\tau} = 0, \qquad \tau < \theta_0. \tag{6.5}$$

Now assume that $\theta > \theta_0$. From Lemma 3 it is not hard to see that there exist $\delta, \varepsilon > 0$ such that

$$\hat{u}^{\tau} \geqslant \delta \hat{u}^{\theta}, \qquad \tau \geqslant \theta - \varepsilon. \tag{6.6}$$

Moreover, we can let $\delta = \delta(\varepsilon)$ tend to 1 as $\varepsilon$ tends to zero. Since $\hat{u}^{\tau} \neq 0$ for $\tau \geqslant \theta - \varepsilon$, with the same reason as before, there exist $\varepsilon_0$, $\delta_0 > 0$ such that

$$\hat{u}^{\theta} \geqslant \delta_0 \hat{u}^{\tau}, \qquad \theta - \varepsilon \leqslant \tau \leqslant \theta + \varepsilon_0. \tag{6.7}$$

When combining (6.6) and (6.7) we obtain that

$$\delta \hat{u}^{\theta} \leqslant \hat{u}^{\tau} \leqslant \frac{1}{\delta_0} \hat{u}^{\theta}, \qquad \theta - \varepsilon \leqslant \tau \leqslant \theta + \varepsilon_0. \tag{6.8}$$

Now if we let $\varepsilon$ and $\varepsilon_0$ tend to zero we can let $\delta$, $\delta_0$ tend to 1 which implies that

$$\lim_{\tau \to \theta} \|\hat{u}^{\tau} - \hat{u}^{\theta}\| = 0, \qquad \theta > \theta_0. \tag{6.9}$$

In view of (6.5) and (6.9) the function $\|\hat{u}^{\theta}\|$ is continuous except possibly at the point $\theta = \theta_0$.

Because of (6.4) we can define

$$p := \lim_{\substack{\theta \to \theta_0 \\ \theta > \theta_0}} \|\hat{u}^{\theta}\|. \tag{6.10}$$

Suppose by contradiction that $p > 0$. Then there exist $\xi(\theta) \in \mathbb{R}$ with

$$\hat{u}^{\theta}(\xi(\theta)) \geqslant p, \qquad \theta > \theta_0.$$

From the equation (5.6) it follows that

$$\hat{u}^{\theta}(t) \geqslant p \exp\left(-\int_{\xi(\theta)}^{t} c(r)\,dr\right), \qquad t \geqslant \xi(\theta),$$

and since all the functions $\hat{u}^{\theta}$ are $\omega$-periodic, there must exist a $q > 0$ such that

$$\hat{u}^{\theta}(s) \geqslant q, \qquad s \in \mathbb{R}, \qquad \theta > \theta_0. \tag{6.11}$$

It is not hard to see that $\theta_0 > 0$, hence, we can choose numbers $\tau$, $\theta$ with $\tau < \theta_0 < \theta$ and $\delta > 0$ such that the inequality (6.1) is fulfilled. But then

$$||\hat{u}^{\tau}|| \geqslant \delta ||\hat{u}^{\theta}|| \geqslant \delta q > 0$$

which is in contradiction to (6.5) and therefore the number $p$ is equal to zero.

Finally, a combined argument of (6.4) and (6.10) shows that $\hat{u}^{\theta_0} = 0$ and the function $||\hat{u}^{\theta}||$ is proved to be continuous.

We summarize our results in the following lemma.

**Lemma 4.** *The function $||\hat{u}^{\theta}||$ is continuous, nondecreasing and strictly increasing if it is positive. Moreover, we have*

$$\lim_{\theta \to \infty} ||\hat{u}^{\theta}|| = 1.$$

*Remark 1.* The assertion in Lemma 4 remains true if we replace $||\hat{u}^{\theta}||$ by the function

$$\Lambda(\hat{u}^{\theta}) = \inf\{\hat{u}^{\theta}(s) \colon s \in \mathbb{R}\}.$$

## 7. Criterion for the Existence of a Positive Periodic Solution

In this chapter we deal with the question whether or not the solution $\hat{u}$ defined in Theorem 2 is positive. We shall give a criterion in terms of the linear equation

$$(L(\theta)) \qquad w'(t) = -c(t)w(t) + \theta J(t, w_t), \qquad t \geqslant 0.$$

**Lemma 5.** *Let $\theta_0 = \inf\{\theta \colon \hat{u}^{\theta} \neq 0\}$. The zero solution of $(L(\theta))$ is asymptotically stable for $\theta < \theta_0$, it is repelling for $\theta > \theta_0$.*

*Proof.* i) Assume that $\theta > \theta_0$ and let $w(t)$ denote a solution of $(L(\theta))$ with $0 \leqslant w_0 \leqslant 1$ and $\chi(w_0) > 0$. Suppose that the function $w(t)$ is bounded, say $|w(t)| \leqslant M$, $t \in \mathbb{R}$. Choose $\delta > 0$ such that

$$\delta |w(t)| \leqslant \tfrac{1}{2}\hat{u}^{\theta}(t), \qquad t \in \mathbb{R}, \tag{7.1}$$

where $\hat{u}^{\theta}$ is the nonzero, $\omega$-periodic solution of $(P(\theta))$. Let $u(t)$ denote the solution of $(P(\theta))$ with $u_0 = \delta w_0$. We observe that the function $\delta w(t)$ also solves the equation $(L(\theta))$. Define

$$\xi = \sup\{t \geqslant 0 \colon \delta w(t) \geqslant u(t)\}$$

and suppose that $\xi < \infty$. Then

$$\delta w'(\xi) - u'(\xi) = \theta J(\xi, \delta w_{\xi}) - \theta(1 - u(\xi))J(\xi, u_{\xi}) > 0,$$

which is obviously a contradiction, therefore

$$u(t) \leqslant \delta w(t), \qquad t \in \mathbb{R}. \tag{7.2}$$

Since the function $w(t)$ is positive we obtain from (7.1) and (7.2) that

$$u(t) \leqslant \tfrac{1}{2}\hat{u}^{\theta}(t), \qquad t \in \mathbb{R}. \tag{7.3}$$

This contradicts Theorem 2 and the zero solution of equation $(L(\theta))$ is proved to be repelling.

ii) Assume now that $\theta < \theta_0$. There exists a $\delta > 0$ such that

$$\theta - \theta_0(1 - \delta) < 0. \tag{7.4}$$

Moreover, since the zero solution of $(P(\theta_0))$ is stable we can find a solution $v(t)$ of $(P(\theta_0))$ with $\inf\{v(s)\colon s \in \mathbb{R}_-\} > 0$ and

$$v(t) \leqslant \delta, \qquad t \geqslant 0. \tag{7.5}$$

From the definition of $\theta_0$ and Theorem 2 it follows that

$$\lim_{t \to \infty} v(t) = 0. \tag{7.6}$$

Consider any solution $w(t)$ of $(L(\theta))$ with $\sup\{|w(s)|\colon s \leqslant 0\} < \infty$. We can find a $p > 0$ such that

$$|w(s)| \leqslant pv(s), \qquad s \in \mathbb{R}_-.$$

Suppose that

$$\xi = \sup\{t \geqslant 0\colon w(t) \leqslant pv(t)\} < \infty,$$

then

$$\begin{aligned} w'(\xi) - pv'(\xi) &= \theta J(\xi, w_\xi) - \theta_0(1 - v(\xi))J(\xi, pv_\xi) \\ &\leqslant J(\xi, pv_\xi)(\theta - \theta_0(1 - v(\xi)). \end{aligned}$$

When using the inequalities (7.4), (7.5) it follows that

$$w'(\xi) - pv'(\xi) < 0,$$

which is impossible. Therefore, we obtain the inequality

$$w(t) \leqslant pv(t), \qquad t \in \mathbb{R}.$$

If we consider $-w$ instead of $w$, it follows that

$$|w(t)| \leqslant pv(t), \qquad t \in \mathbb{R},$$

and from (7.6) we see that the function $w(t)$ tends to zero as $t$ tends to infinity. □

In a forthcoming paper [13] a criterion for the existence of nontrivial periodic solutions is given.

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Received May 5/Revised June 1, 1982