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Stability Conditions for Systems of Linear Nonautonomous Delay Differential Equations

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Systems of linear nonautonomous delay differential equations are considered which are of the form $y_i'(t)=\sum_{k=1}^{n}\int_0^T b_{ik}(t,\ s)\ y_k(t-s)\ d\eta_{ik}(s)-c_i(t)\ y_i(t)$, where $i=1,\dots,n$. Sufficient conditions are derived for both the asymptotic stability and the instability of the zero solution. The main result is found by a monotone technique using elementary methods only. Moreover, additional criteria are obtained by using the method of Lyapunov functionals. © 1986 Academic Press, Inc.

1. INTRODUCTION

Delay differential equations often occur in Mathematical Biology, Ecology, and many other areas, whenever time lags are taken into consideration. A variety of such applications are presented in [2].

One of the main problems here is to decide whether or not there exist equilibrium solutions, and if they do, whether or not they are stable. The local stability problem often depends on the linearized part of the differential equation around the equilibrium solution. For a typical example of such a situation see [7]. This is why we can restrict our stability analysis to linear systems.

We consider systems of the form

$$y_i'(t)=\sum_{k=1}^{n}\int_0^T b_{ik}(t,s)\ y_k(t-s)\ d\eta_{ik}(s)-c_i(t)\ y_i(t), \tag{1}$$

where $t\geqslant 0$, $0\leqslant T<\infty$, $i=1,\dots,n$. The functions $b_{ik}(t,s)$ and $c_i(t)$ are assumed to be continuous on $\mathbb{R}_+\times[0,T]$ and $\mathbb{R}_+$, respectively. Moreover, the functions $\eta_{ik}(s)$ are assumed to be nondecreasing on $[0,T]$. A special case of (1) occurs when $n=1$ and $\eta_{11}(s)$ is a piecewise constant function with jumps of length 1 at $T_j\leqslant T$.

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In this case the equation takes the form

$$y_1'(t) = \sum_{j=1}^{p} b_{11}(t, T_j)\, y_1(t - T_j) - c_1(t)\, y_1(t). \tag{2}$$

The last equation has already been studied by several other authors. They derive sufficient conditions for the zero solution to be asymptotically stable. See [3; 4, p. 108]. In the present paper, not only sufficient, but also necessary stability conditions are derived. The results in Section 2 are obtained by a monotone technique which requires elementary methods only. These results are applied to a special case in Section 3. Finally, in Section 4 we use Lyapunov functionals in a manner similar to that in [3, 4, 5, 6] to prove additional criteria. The idea of the proof is similar to that in [6], however, it is applied to a more general equation. We use linear Lyapunov functionals and this case is easier to handle than the case of quadratic Lyapunov functionals as it was done in [3]. The reader can see [1] for a different type of necessary and sufficient conditions for stability of certain systems.

It is mentioned that the criteria obtained in this paper are actually useful in practice as is illustrated by some examples.

2. Basic Criteria

We consider the system (1) and assume that the functions b_{ik}, c_i are continuous and that the functions η_{ik} are nondecreasing, $i, k = 1, \ldots, n$. It is well known that a solution of (1) exists on $\mathbb{R}_+$ and is uniquely determined by preassigning its values for $t \in [-T, 0]$. Moreover, the solution has the representation

$$\begin{aligned} y_i(t) &= e^{-\int_0^t c_i(r)\,dr}\, y_i(0) \\ &\quad + \int_0^t e^{-\int_x^t c_i(r)\,dr} \left(\sum_{k=1}^{n} \int_0^T b_{ik}(x, s)\, y_k(x - s)\, d\eta_{ik}(s) \right) dx, \end{aligned} \tag{3}$$

where $i, k = 1, \ldots, n$ and $t \geqslant 0$.

To establish criteria for the stability of the system (1) we also consider the system

$$z_i'(t) = \sum_{k=1}^{n} \int_0^T |b_{ik}(t, s)|\, z_k(t - s)\, d\eta_{ik}(s) - c_i(t)\, z_i(t), \tag{4}$$

where $i, k = 1, \ldots, n$ and $t \geqslant 0$. Clearly, the equations in (3) also hold if we replace y_i and b_{ik} by z_i and $|b_{ik}|$, respectively. We will make use of this in

LEMMA 1. *Let $y_i(t)$ and $z_i(t)$ be solutions of the systems* (1) *and* (4), *respectively, such that*

$$y_i(t) < z_i(t), \qquad i = 1, ..., n, \tag{5}$$

for $t \in [-T, 0]$. Then the inequalities (5) *also hold for $t \geqslant 0$.*

Proof. Define the function $\psi(t) = \min_i(z_i(t) - y_i(t))$, $t \geqslant -T$. It is easy to see that $\psi(t)$ is continuous and $\psi(t) > 0$ for $t \in [-T, 0]$. Now suppose that the assertion of the lemma is wrong. Then there exists a number $\xi > 0$ such that $\psi(t) > 0$ for $t \in [-T, \xi)$ and $\psi(\xi) = 0$. On the other hand it follows from (3) that

$$\begin{aligned} z_i(\xi) - |y_i(\xi)| \geqslant\; & e^{-\int_0^\xi c_i(r)\,dr}(z_i(0) - |y_i(0)|) \\ & + \int_0^\xi e^{-\int_x^\xi c_i(r)\,dr} \\ & \times \sum_{k=1}^n \int_0^T |b_{ik}(x, s)\, z_k(x-s) - b_{ik}(x, s)\, y_k(x-s)|\, d\eta_{ik}(s)\, dx, \end{aligned}$$

for all $i = 1, ..., n$. From our assumptions it is easy to see that the second term on the right-hand side of the above expression must be greater or equal to zero. Hence, we have that

$$z_i(\xi) - |y_i(\xi)| \geqslant e^{-\int_0^\xi c_i(r)\,dr}(z_i(0) - |y_i(0)|) > 0,$$

for all $i = 1, ..., n$. Therefore, we conclude that $\psi(\xi) > 0$ and obtain a contradiction to the choice of ξ and the proof of the lemma is finished.

An easy consequence of Lemma 1 is the following result.

COROLLARY 1. The zero solution of the system (1) is (asymptotically) stable provided that the zero solution of the system (4) is (asymptotically) stable.

This allows us to restrict our stability analysis to the system (4). If we choose $y_i(t) = 0$ in Lemma 1 we obtain

COROLLARY 2. *Suppose that $z_i(t)$ is a solution of* (4) *with $z_i(t) > 0$ for $t \in [-T, 0]$ and $i = 1, ..., n$. Then $z_i(t) > 0$ for all $t \geqslant -T$ and $i = 1, ..., n$.*

COROLLARY 3. *Assume that $z_i^{(1)}$, $z_i^{(2)}$ are both solution of* (4). *If the inequalities $z_i^{(1)}(t) > z_i^{(2)}(t)$, $i = 1, ..., n$, hold for all $t \in [-T, 0]$ then they also hold for all $t \geqslant -T$.*

Proof. Since we are dealing with a linear equation we have that $z_i(t) = z_i^{(1)}(t) - z_i^{(2)}(t)$ is a solution of (4), too. Now the result follows easily from Corollary 2.

Now we can present our basic criteria.

THEOREM 1. *The system* (4) *is asymptotically stable if and only if there exist constants* $t_0 \geqslant 0$, $\alpha_i > 0$ *and continuous functions* $g_i(t)$ *such that the conditions*

$$\lim_{s \to \infty} \int_{t_0}^{s} g_i(r)\, dr = -\infty, \tag{6}$$

$$\sum_{k=1}^{n} \int_0^T |b_{ik}(t, s)|\, \alpha_k e^{\int_{t_0}^{t-s} g_k(r)\, dr}\, d\eta_{ik}(s)$$
$$\leqslant (c_i(t) + g_i(t))\, \alpha_i e^{\int_{t_0}^{t} g_i(r)\, dr} \tag{7}$$

hold for all $t \geqslant t_0$ *and* $i = 1, ..., n$. *Conditions* (6) *and* (7) *are sufficient for asymptotic stability of* (1).

Proof. (i) Suppose that the system (4) is asymptotically stable. From Corollary 2 it is easy to see that there must exist a solution $z_i(t)$ of (4) with $z_i(t) > 0$ for $t \geqslant -T$. If we define $\alpha_i = z_i(0)$ and $g_i(t) = z_i'(t)/z_i(t)$ then the following identities hold:

$$z_i(t) = \alpha_i e^{\int_0^t g_i(r)\, dr}, \qquad i = 1, ..., n.$$

Now it is easy to see that equality holds in (7) with $t_0 = 0$, and (6) holds as well since $\lim_{t \to \infty} z_i(t) = 0$.

(ii) Suppose that the statements (6), (7) hold for some functions $g_i(t)$ and some $\alpha_i > 0$. For $\varepsilon > 0$ define

$$w_i(t, \varepsilon) = \alpha_i e^{\int_{t_0}^{t} (g_i(r) + \varepsilon)\, dr}, \qquad i = 1, ..., n.$$

Now consider a solution $z_i(t)$ of (4) with $z_i(t) > 0$ for $t \in [-T, 0]$ and $i = 1, ..., n$. There must exist a number $\mu > 0$ such that

$$\mu z_i(t) < w_i(t, 0), \qquad t \in [t_0, t_0 + T],\ i = 1, ..., n.$$

Since we are dealing with a linear equation we have that $\mu z_i(t)$ is a solution of (4), too, and we can assume without loss of generality that $\mu = 1$. Next, suppose that there would exist $\varepsilon > 0$ and a minimally chosen $\xi > t_0 + T$ such that for some index j the equation $w_j(\xi, \varepsilon) = z_j(\xi)$ would hold. Then $z_i(t) < w_i(t, \varepsilon)$ for all $t \in [\xi - T, \xi)$ and $i = 1, ..., n$. On the other hand, since $w_j(\xi, \varepsilon) = z_j(\xi)$ we obtain from (4) and (7) that

$$\begin{aligned} z_j'(\xi) &= \sum_{k=1}^{n} \int_0^T |b_{jk}(\xi, s)| \, z_k(\xi - s) \, d\eta_{jk}(s) - c_j(\xi) \, w_j(\xi, \varepsilon) \\ &\leqslant \sum_{k=1}^{n} \int_0^T |b_{jk}(\xi, s)| \, w_k(\xi - s, \varepsilon) \, d\eta_{jk}(s) - c_j(\xi) \, w_j(\xi, \varepsilon) \\ &\leqslant \sum_{k=1}^{n} \int_0^T |b_{jk}(\xi, s)| \, e^{(\xi - t_0 - s)\varepsilon} w_k(\xi - s, 0) \, d\eta_{ik}(s) \\ &\quad - c_j(\xi) \, e^{(\xi - t_0)\varepsilon} w_j(\xi, 0) \\ &\leqslant g_j(\xi) \, e^{(\xi - t_0)\varepsilon} w_j(\xi, 0) < (g_j(\xi) + \varepsilon) \, w_j(\xi, \varepsilon) = w_j'(\xi, \varepsilon). \end{aligned}$$

Hence, we have that $z_j'(\xi) < w_j'(\xi, \varepsilon)$ and this is obviously in contradiction to the choice of ξ, ε. Therefore, the inequalities $z_i(t) \leqslant w_i(t, \varepsilon)$ hold for all $t \geqslant t_0$, $\varepsilon > 0$, and $i = 1,..., n$. Using (6) it follows that $\lim_{t \to \infty} z_i(t) = 0$. So we have found one solution which tends to the zero solution. Now given any solution $\tilde{z}_i(t)$ of (4). There must exist a number $\theta > 0$ such that $-z_i(t) < \theta \tilde{z}_i(t) < z_i(t)$ for all $t \in [-T, 0]$ and $i = 1,..., n$. Now from Corollary 3 we can see that $\lim_{t \to \infty} \theta z_i(t) = 0$ and the zero solution is proved to be asymptotically stable.

Using the same method as before one can establish the following counterpart to Theorem 1.

THEOREM 2. *The system* (4) *has unbounded solutions if and only if there exist constants* $t_0 \geqslant 0$, $\alpha_i > 0$ *and continuous functions* g_i *such that*

$$\limsup_{s \to \infty} \int_{t_0}^{s} g_j(r) \, dr = \infty \qquad \text{for some} \quad j \in \{1,..., n\}, \tag{8}$$

$$\begin{aligned} &\sum_{k=1}^{n} \int_0^T |b_{ik}(t, s)| \, \alpha_k e^{\int_{t_0}^{t-s} g_k(r) \, dr} \, d\eta_{ik}(s) \\ &\qquad \geqslant (c_i(t) + g_i(t)) \, \alpha_i e^{\int_{t_0}^{t} g_i(r) \, dr} \end{aligned} \tag{9}$$

for all $t \geqslant t_0$ *and* $i = 1,..., n$.

It is mentioned that the system (1) may be asymptotically stable even though the system (4) may not be asymptotically stable.

COROLLARY 4. (i) *The systems* (1), (4) *are asymptotically stable provided that there exist constants* $t_0 \geqslant 0$, $\varepsilon > 0$, $\alpha_i > 0$ *such that*

$$e^{\varepsilon T} \sum_{k=1}^{n} \int_0^T \alpha_k \, |b_{ik}(t, s)| \, d\eta_{ik}(s) \leqslant \alpha_i (c_i(t) - \varepsilon) \tag{10}$$

for all $t \geqslant t_0$ and $i = 1,..., n$.

(ii) *The system* (4) *has unbounded solutions if there exist numbers* $t_0 \geqslant 0$, $\varepsilon > 0$, $\alpha_i > 0$ *such that*

$$e^{-\varepsilon T} \sum_{k=1}^{n} \int_0^T \alpha_k \, |b_{ik}(t, s)| \, d\eta_{ik}(s) \geqslant \alpha_i(c_i(t) + \varepsilon) \tag{11}$$

for all $t \geqslant t_0$ *and* $i = 1, ..., n$.

Proof. To prove (i) we put $g_i(t) = -\varepsilon$ and show that (7) holds. But this follows easily from (10) and the fact that

$$e^{\int_{t_0}^{t-s}(-\varepsilon)\, dr} \leqslant e^{\varepsilon T} e^{\int_{t_0}^{t}(-\varepsilon)\, dr}, \qquad s \in [0, T].$$

The proof of (ii) follows analogously from Theorem 2 with $g_i(t) = \varepsilon$.

The applicability of Corollary 4 is demonstrated in

EXAMPLE 1. Consider the system

$$y_1'(t) = \cos(t)\, y_1(t-1) + t^2 y_2(t) - e^t y_1(t),$$

$$y_2'(t) = \int_0^{2\Pi} \sin(t-s)\, y_1(t-s)\, ds + \sum_{m=3}^{7} y_2(t-m) - 6 y_2(t).$$

In this case $T = 7$ and the inequalities (10) become

$$e^{7\varepsilon}(\alpha_1 \, |\cos(t)| + \alpha_2 t^2) \leqslant \alpha_1(e^t - \varepsilon),$$

$$e^{7\varepsilon}\left(\alpha_1 \int_0^{2\Pi} |\sin(t-s)| \, \mathrm{ds} + \alpha_2 5\right) \leqslant \alpha_2(6 - \varepsilon).$$

For example, we can choose $\alpha_1 = 1$, $\alpha_2 = 10$, and $\varepsilon = 10^{-4}$. Then the first inequality holds if t is big enough, and from the fact that $e^{7\varepsilon}(2\Pi + 50) \leqslant 10(6 - \varepsilon)$ it is easy to see that the second inequality holds, too. Hence, the system is shown to be asymptotically stable.

3. SCALAR EQUATIONS WITH A SINGLE DELAY

This section is devoted to the study of a very special case of (1), namely, we put $n = 1$ and $\eta_{11}(s) = 0$ for $s < T$ and $\eta_{11}(s) = 1$ for $s \geqslant T$. Moreover, we require that $b_{11}(t, s)$ does not depend on s. When omitting the subscripts this leads to the following equation

$$y'(t) = b(t)\, y(t-T) - c(t)\, y(t), \qquad t \geqslant 0. \tag{12}$$

This equation has already been treated in [3]. One of the results obtained in this paper says that the zero solution is asymptotically stable provided that $b(t)$ is periodic of period T and there exists $q>0$ such that

$$|b(t)| \leqslant c(t)-q, \qquad t \geqslant 0. \tag{13}$$

This result is also an easy consequence of Corollary 4. Namely, from the fact that $b(t)$ is bounded and (13) it follows that there must exist $\varepsilon>0$ such that

$$e^{\varepsilon T}\,|b(t)| \leqslant c(t)-\varepsilon, \qquad t \geqslant 0.$$

Observe, that we did not make use of the assumption that $b(t)$ is periodic. It is enough to assume that $b(t)$ is bounded. Now we specialize Theorem 1 to Eq. (12).

COROLLARY 5. *The zero solution of* (12) *is asymptotically stable provided that there exists* $t_0 \geqslant 0$ *and a continuous function* $g(t)$ *such that*

$$\lim_{s\to\infty} \int_{t_0}^{s} g(r)\,dr = -\infty, \tag{14}$$

$$|b(t)|\, e^{-\int_{t-T}^{t} g(r)\,dr} \leqslant c(t)+g(t), \qquad t \geqslant t_0. \tag{15}$$

In the special case that $T=0$, i.e., there is no delay, we can choose $g(t)=|b(t)|-c(t)$. Then we actually have equality in (15). In fact, the zero solution is asymptotically stable if

$$\lim_{s\to\infty} \int_{0}^{s} (|b(r)|-c(r))\,dr = -\infty. \tag{16}$$

Now the question arises if this is also true in the case where $T>0$. The answer is negative as the following example shows. Indeed, there exist $T>0$ and functions $b(t)$, $c(t)$ such that the zero solution is asymptotically stable and

$$\lim_{s\to\infty} \int_{0}^{s} (|b(r)|-c(r))\,dr = \infty. \tag{17}$$

EXAMPLE 2. For a fixed number $u>0$ define the functions

$$b(t)=\begin{cases} u^2\left(\frac{1}{2}-\left|\frac{5}{2}-t\right|\right), & t\in[2,3) \\ 0, & t\in[0,2)\cup[3,10), \end{cases}$$

$$g(t)=\begin{cases} u(-1+2t), & t\in[0,1) \\ u, & t\in[1,4) \\ u(9-2t), & t\in[4,5) \\ -u, & t\in[5,10) \end{cases}$$

and extend these functions periodically on all of $\mathbb{R}_+$. Moreover, we put $c(t)=u$ and $T=1$. Now we have that

$$\int_0^{10} |b(r)|\,dr-\int_0^{10} c(r)\,dr=u^2\,\tfrac{1}{4}-10u.$$

Hence, if we require that $u>40$ and since $b(t)$, $c(t)$ are periodic it follows that (17) holds. Furthermore, when integrating the function $g(t)$ over the four different ranges specified above, we find that

$$\int_0^{10} g(r)\,dr=u(0+3+0-5)=-2u<0$$

and hence, the Eq. (14) must hold. Inequality (15) is certainly satisfied if $t\in[0,2)\cup[3,10)$. In the case that $t\in[2,3)$ we observe that $\int_{t-1}^{t} g(r)\,dr=u$ and

$$|b(t)|\,e^{-\int_{t-1}^{t} g(r)\,dr}\leqslant \tfrac{1}{2}u^2e^{-u}.$$

Moreover, in this case $g(t)+c(t)=2u$ and we can find $u>40$ such that

$$\tfrac{1}{2}u^2e^{-u}\leqslant 2u.$$

For this choice of u the inequality (15) is satisfied for all $t\geqslant 0$ and the zero solution is proved to be asymptotically stable.

Remark. Analogously as in Example 2 one can find functions $b(t)$, $c(t)$ such that the zero solution is not asymptotically stable and (16) is fulfilled.

4. Linear Lyapunov Functionals

In Theorems 1 and 2 we have established criteria for the asymptotic stability of the zero solution which involve summations over the second index of the matrices (b_{ik}) and (η_{ik}). In this section we derive additional criteria which involve summations over the first index of the matrices.

Theorem 3. *The systems* (1) *and* (4) *are asymptotically stable if there exist constants* $t_0\geqslant 0$, $\alpha_i>0$ *and continuous functions* $g_i(t)$ *such that*

$$\lim_{s \to \infty} \int_{t_0}^{s} g_i(r)\, dr = \infty, \tag{18}$$

$$\sum_{k=1}^{n} \int_0^T |b_{ki}(t+s, s)|\, \alpha_k e^{\int_{t_0}^{t+s} g_k(r)\, dr}\, d\eta_{ki}(s) \\ \leqslant (c_i(t) - g_i(t))\, \alpha_i e^{\int_{t_0}^{t} g_i(r)\, dr} \tag{19}$$

hold for all $t \geqslant t_0$ *and* $i = 1,..., n$.

Proof. Let $z_i(t)$ be a solution of (4) with $z_i(t) > 0$ for $t \in [-T, 0]$ and $i = 1,..., n$. From Corollary 2 we know that $z_i(t) > 0$ for $t \geqslant -T$ and $i = 1,..., n$. Now we put

$$h_i(t) = \alpha_i e^{\int_{t_0}^{t} g_i(r)\, dr}$$

and introduce the linear Lyapunov functional

$$w(t) = \sum_{i=1}^{n} h_i(t)\, z_i(t) \\ + \sum_{i,k=1}^{n} \int_0^T \int_{t-s}^{t} h_i(r+s)\, |b_{ik}(r+s, s)|\, z_k(r)\, dr\, d\eta_{ik}(s).$$

From (4) it follows that

$$w'(t) = \sum_{i=1}^{n} h_i'(t)\, z_i(t) \\ + \sum_{i,k=1}^{n} h_i(t) \int_0^T |b_{ik}(t, s)|\, z_k(t-s)\, d\eta_{ik}(s) \\ - \sum_{i=1}^{n} h_i(t)\, c_i(t)\, z_i(t) \\ + \sum_{i,k=1}^{n} \int_0^T h_i(t+s)\, |b_{ik}(t+s, s)|\, z_k(t)\, d\eta_{ik}(s) \\ - \sum_{i,k=1}^{n} h_i(t) \int_0^T |b_{ik}(t, s)|\, z_k(t-s)\, d\eta_{ik}(s).$$

The second and the last term in the expression above cancel and after rearranging the order of summation we obtain

$$w'(t) = \sum_{i=1}^{n} \Bigg(h_i'(t) - h_i(t)\, c_i(t) \\ + \sum_{k=1}^{n} \int_0^T h_k(t+s)\, |b_{ki}(t+s, s)|\, d\eta_{ki}(s) \Bigg) z_i(t).$$

Now from (19) it follows tat

$$h_i'(t) - h_i(t)\, c_i(t) + \sum_{k=1}^{n} \int_0^T h_k(t+s)\, |b_{ki}(t+s, s)|\, d\eta_{ki}(s) \leqslant 0$$

for all $t \geqslant t_0$ and $i = 1,..., n$. Since $z_i(t) > 0$ it follows that $w'(t) \leqslant 0$ for $t \geqslant t_0$ and therefore, the function $w(t)$ is bounded from above.

From the definition of $w(t)$ it is easy to see that

$$0 \leqslant h_i(t)\, z_i(t) \leqslant w(t), \qquad t \geqslant t_0,\ i = 1,..., n. \tag{20}$$

Moreover, we can see from (18) that

$$\lim_{t \to \infty} h_i(t) = \infty, \qquad i = 1,..., n,$$

and hence, from (20) it follows that

$$\lim_{t \to \infty} z_i(t) = 0, \qquad i = 1,..., n.$$

So we have found one solution which tends to the zero solution and the proof finishes in the same way as the proof of Theorem 1.

Similarly, one obtains the following counterpart to Theorem 3. However, one needs the additional assumption that all the functions b_{ik} are bounded to make the analogous conclusion as it was done in (20). Namely, from the conditions

$$\lim_{s \to \infty} h_i(s) = 0, \qquad i = 1,..., n,$$

$$w(t) \geqslant \delta > 0, \qquad t \geqslant t_0,$$

we find that

$$\limsup_{s \to \infty} z_j(s) = \infty$$

for some $j \in \{1,..., n\}$.

THEOREM 4. *Suppose that the functions b_{ik} are bounded and there exist constants $t_0 \geqslant 0$, $\alpha_i > 0$ and continuous functions $g_i(t)$ such that the conditions*

$$\lim_{s \to \infty} \int_{t_0}^{s} g_i(r)\, dr = -\infty, \tag{21}$$

$$\sum_{k=1}^{n} \int_0^T |b_{ki}(t+s, s)| \, \alpha_k e^{\int_{t_0}^{t+s} g_k(r)\, dr} \, d\eta_{ki}(s)$$
$$\geqslant (c_i(t) - g_i(t)\, \alpha_i e^{\int_{t_0}^{t} g_i(r)\, dr} \tag{22}$$

hold for all $t \geqslant t_0$ and $i = 1, ..., n$. Then the system (4) *has unbounded solutions.*

One can derive the following result from Theorem 3 and 4 in a way analogous to the proof of Corollary 4.

COROLLARY 6. (i) *The systems* (1) *and* (4) *are asymptotically stable provided that there exist $t_0 \geqslant 0$, $\varepsilon > 0$, $\alpha_i > 0$ such that*

$$e^{\varepsilon T} \sum_{k=1}^{n} \int_0^T \alpha_k \, |b_{ki}(t+s, s)| \, d\eta_{ki}(s) \leqslant \alpha_i (c_i(t) - \varepsilon) \tag{23}$$

for all $t \geqslant t_0$ and $i = 1, ..., n$.

(ii) *If the functions b_{ik} are bounded and there exist $t_0 \geqslant 0$, $\varepsilon > 0$, $\alpha_i > 0$ such that*

$$e^{-\varepsilon T} \sum_{k=1}^{n} \int_0^T \alpha_k \, |b_{ki}(t+s, s)| \, d\eta_{ki}(s) \geqslant \alpha_i (c_i(t) + \varepsilon) \tag{24}$$

for all $t \geqslant t_0$ and $i = 1, ..., n$, then the system (4) *has unbounded solutions.*

The applicability of Corollary 6 is illustrated by the following example.

EXAMPLE 3. We consider the system

$$z_1'(t) = \int_0^{\Pi} (1 + \cos(t-s)) \, z_1(t-s) \, ds + \tfrac{1}{2} z_2(t-4) - 2 z_1(t),$$

$$z_2'(t) = 3 z_1(t) + \int_1^2 z_2(t-s) \, ds - z_2(t).$$

In this case we can choose $T = 4$ and the inequalities (24) become

$$e^{-4\varepsilon} \alpha_1 (1 + \cos(t)) \Pi + 3\alpha_2 \geqslant \alpha_1 (2 + \varepsilon)$$
$$e^{-4\varepsilon} \alpha_1 \tfrac{1}{2} + \alpha_2 \leqslant \alpha_2 (1 + \varepsilon).$$

Now we can choose, e.g., $\alpha_1 = \alpha_2 = 1$ and the inequalities above certainly hold for sufficiently small $\varepsilon > 0$ and the system is proved to have unbounded solutions.

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